CHAPTER TWO

BRAIN WAVES AS RESONANCE

2.1 THE PHENOMENON OF STANDING WAVES

When two sinusoidal waves of the same wavelength and amplitude are traveling in the opposite directions, they superpose and produce *standing waves* (Halliday, Resnick 1988).

Consider two transverse sinusoidal waves in the xy-plane. One of them is traveling in the positive x-direction and the other in the negative x-direction. They are represented as

(2.1a) $$y_{+}(x,t) = y_m \sin(kx - \omega t)$$

(2.2b) $$y_{-}(x,t) = y_m \sin(kx + \omega t)$$

y_m is the amplitude, k and ω are the wavenumber and angular frequency respectively. No phase constants are introduced because it means nothing to speak of a phase difference between waves that travel in opposite directions. The principle of superposition gives, for the combined wave

$$y(x,t) = y_{+}(x,t) + y_{-}(x,t)$$

Using $\sin\alpha + \sin\beta = 2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)$, we have

(2.2) $$y(x,t) = [\,2y_m \sin kx\,]\cos\omega t$$

This equation is not of the form of a traveling wave. It, in fact, describes a *standing wave*. The absolute value of quantity in square brackets of eq. (2.2) is the amplitude of oscillation of the string element located at position x.

In a traveling sinusoidal wave, the oscillation amplitude is the same for all string elements, regardless of their location. However, that is not the case for a standing wave in which there are certain values of x for which the amplitude is zero. These are called *nodes*. There are also values of x for which the amplitude of vibration has its maximum value, that is, $2y_m$. These are called *antinodes*. From eq. (2.2), we can write the conditions for nodes and antinodes as

(2.3a) $$x = n\,\lambda/2 \quad \text{(nodes)}$$

(2.3b) $$x = (n + \tfrac{1}{2})\,\lambda/2 \quad \text{(antinodes)}$$

where $n = 0,1,2,3,\ldots$.

We can set up standing waves in a system at certain specified frequencies. We say that the system *resonates* at these frequencies.

Consider a string of length L clamped at each end. Because the ends of the string cannot move, a node of the standing wave pattern must exist at each end of the string. The length of the string must, then, be an integral multiple of a half-wavelength or $L = n(\lambda/2)$. The allowed wavelengths are

(2.4a) $$\lambda = 2L/n$$

where $n = 1,2,3\ldots\ldots$ The allowed frequencies are

(2.4b) $$\nu = v/\lambda = n(v/2L)$$

Only if the stretched string is wiggled at one of the frequencies given by eq. (2.4b) will a pattern of standing waves develop.

If the string is wiggled at a frequency not given by eq. (2.4b), it will not be possible to transfer energy efficiently from the external wiggling agent to the string. For some time intervals. the external agent will do work on the string; at other intervals the string will do work on the vibrator. At *resonance* the energy flow is entirely from the vibrator to the string. The amplitude of vibration builds up until the vibrating string loses energy by friction like losses just as fast as it receives energy from the vibrator.

The oscillation frequency of a stretched string is quantized because the string is not infinitely long and thus has two ends. Each end should be a node if the string is fixed at the two ends. An infinitely long string (no ends) could indeed support a standing wave pattern at any frequency.

All oscillating systems that are localized in space oscillate only at discrete frequencies. Requiring that wave vanish at the boundaries of the system leads to frequency quantization.

A stretched string has many resonant frequencies whereas a mass-spring system has only one. The reason is that, in the mass-spring system, the inertia of the system is concentrated into one system element (tho mass) and the elasticity into another (the spring). On the contrary, in a stretched string the inertia and the elasticity are spread uniformly throughout the system, every element of the string having some of each. In the first case, there is only one possible mechanical arrangement that permits energy to flow back and forth from kinetic to potential forms as the system oscillates. In the second case, there are many such arrangements, each with its own resonant frequency.

2.2 STANDING WAVES IN PHYSICAL SYSTEMS

The phenomenon of resonance is common to all oscillating systems. The discrete frequencies of the standing waves in guitar string (one-dimensional system), the many oscillation modes of a kettledrum (two-dimensional system), the stationary states in which atoms may exist (three-dimensional oscillation modes of the matter waves that represent the atomic electrons) are all examples of systems that are localized in space.

The phenomenon of standing waves was used by Planck to arrive at the correct expression of energy density of black-body radiation. Consider a photon of frequency ν propagating in a direction $\mathbf{n}$ inside a box. The wavevector of the photon is $\mathbf{k} = (2\pi/\lambda)\mathbf{n} = (2\pi\nu/c)\mathbf{n}$. If each dimension of the box L_x, L_y, and L_z is much larger than a wavelength, then the photon can be represented by some sort of standing wave in the box (Rybicki, Lightman 1979). The number of nodes in the wave in each direction x, y, z is $n_x = k_x L_x/2\pi$, since there is one node for each integral number of wavelengths in given orthogonal directions. When the number of nodes in a given direction changes by one or more, the wave can be said to have changed states in a distinguishable manner. If $n_i \geq 1$, we can write the number of node changes in a wave number interval as

$$\Delta n_x = L_x \, \Delta k_x/2\pi \quad (2.5)$$

The number of states in the three-dimensional wave vector element $d^3k = \Delta k_x \Delta k_y \Delta k_z$ is

$$(2.6) \qquad \Delta N = \Delta n_x \Delta n_y \Delta n_z = L_x L_y L_z d^3k/(2\pi)^3$$

Since $L_x L_y L_z = V$, the volume of the box, we note that the number of states per unit volume per unit three-dimensional wave number is $2/(2\pi)^3$. The factor 2 appears because photons have two independent polarizations (two states per unit wave vector **k**). Using the expression

$$d^3k = k^2 dk d\Omega = (2\pi)^3 \nu^2 d\nu d\Omega / c^2$$

we find the density of states (the number of states per unit solid angle, $d\Omega$, per unit volume per unit frequency) to be

$$(2.7) \qquad \rho_s = 2\nu^2/c^3$$

To find the average energy of each state, we note that each state may contain n photons of energy $h\nu$, where $n = 0,1,2,...$ Thus the energy may be $E_n = nh\nu$. according to statistical mechanics, the probability of a state of energy E_n is proportional to $exp(-\beta E_n)$ where $\beta = 1/kT$ and k = Boltzmann's constant. The average energy may, therefore, be written as

$$(2.8) \qquad \langle E \rangle = \frac{\sum_{n=0}^{\infty} E_n e^{-\beta E_n}}{\sum_{n=0}^{\infty} e^{-\beta E_n}}$$

Summation of the geometric series gives

$$(2.9) \qquad \langle E \rangle = \frac{h\nu}{exp(h\nu/kT) - 1}$$

Since $h\nu$ is the energy of one photon of frequency ν, eq. (2.9) says that the average number of photons of frequency ν is given by

$$n_\nu = [\ exp(h\nu/kT) - 1\ ]^{-1} \tag{2.10}$$

where n_ν is the *occupation number*. The energy per unit solid angle per unit volume per unit frequency is the product of $\langle E \rangle$ and the density of states given in eq. (2.7). Therefore, the energy density of blackbody radiation is

$$u_\nu(\Omega) = \frac{2h\nu^3/c^2}{exp(h\nu/kT) - 1} \tag{2.11}$$

This is *Planck's law of blackbody radiation.*

The standing wave concept is also helpful to explain why the angular momentum is quantized in Bohr atom. Consider the vibrations of a wire loop. If the circumference of the loop of wire is an integral multiple of λ, the wavelength fits into the loop's circumference, each wave joining smoothly with the next. If this condition is not satisfied, the vibrations will die out rapidly. By treating the behavior of electron waves in the hydrogen atom similar to the vibrations of a wire loop, we can explain why the electron is stable in certain orbits. If the circumference of the orbit is an integral multiple of λ, standing waves are set up and maintained indefinitely i.e.

$$2\pi r = n\lambda;\ n = 1,2,3...... \tag{2.12}$$

Putting $\lambda = h/mv$ (De Broglie's expression) we get $2\pi r = nh/mv$, which gives

$$mvr = n\ h/2\pi \tag{2.13}$$

which is Bohr's angular momentum quantization condition.

Standing wave concept is also being used in *superstring models* (Schwarz 1986) which are possible candidates for a unified theory of all forces of nature.

2.3 ELECTROENCEPHALOGRAM AS A SYSTEM OF STANDING WAVES

The brain is a spatially localized system. We, therefore, expect the existence of some discrete frequencies whose superposition may be recorded as *electroencephalogram* (EEG). Nunez (1981) has described a model treating brain waves as resonance. The model is based on the following assumptions:

(*i*) There is a natural frequency of the system which is a function of the circumference of the cortical pathway.

(*ii*) The voltage in the dendritic tree behaves as damped harmonic oscillator. The damping present in the system produces nonlinearity.

(*iii*) Only those frequencies of (*ii*) are observed which match with (*i*).

(*iv*) The superposition of all the frequencies of (*iii*) is the EEG. It is assumed that the system is underdamped and hence the nonlinearity produced is small making it possible to apply the superposition principle.

(*v*) When brain is in a certain rhythm, most of the oscillators oscillate with a characteristic frequency ω_{rhythm} (Cramer's Central Limit Theorem). The variance is finite and $\sigma_{\omega} \ll 1$.

In the one-dimensional, homogeneous picture, the wavenumbers are restricted to the discrete values $k_n = 2n\pi/L$; n = 1,2,3,.... where L is the circumference of the cortical pathway. The requirement that both the potential and its first derivative be continuous limits k_n differently than in the case of a stretched string with its ends fixed. For example, in the case of waves traveling along the midline, around the frontal pole, inside the temporal pole, and back around the occipital pole, direct measurement yields a circumference of 45 cm. In order to account for infolding of the cortical surface, this figure must be multiplied by a fqactor of 2.2, so that this estimate yields

$$L \simeq 100 \text{ cm}, \; k_1 \simeq 0.06 \text{ cm}^{-1}$$

This value of the wavenumber k lies approximately at the lower end of the range of the measured midline wavenumber spectrum of the alpha rhythm (Nunez 1974; Nunez, Allen, Bickford 1978).

The model explains rhythms and frequency relationships. Simulation may predict statistical weight, frequency distribution and statistics of the system (Siddiqui, Kamal, Khan 1988).