Chapter 5

Modified Murthy Estimators

5.1 Introduction:

In this chapter a series of new estimators has been developed following the idea of Murthy (1957). These estimators have been developed by using different selection procedures in the general Murthy estimator given as:

$$t_{symm} = \frac{1}{P(S)} \sum_{i=1}^{n} P(S|i) y_i \tag{1.5.10}$$

Murthy (1957) used the Yates – Grundy draw – by – draw procedure in estimator (1.5.10) to obtain following unbiased estimator of population total for a sample of size 2:

$$t_{symm} = \frac{\left[\frac{y_i}{p_i}(1 - p_j) + \frac{y_j}{p_j}(1 - p_i)\right]}{(2 - p_i - p_j)} \tag{1.5.11}$$

Three new estimators have been developed by using various selection procedures in the estimator given in equation (1.5.10). A fourth estimator has been proposed following the idea of Raj (1956a).

5.2 Modified Murthy Estimators:

In this section and the following sub – sections some new estimators has been developed that can be used with unequal probability sample without replacement to estimate the population total. These estimators have been developed by using Brewer, Durbin and New selection procedure – III in the general Murthy estimator given in equation (1.7.10).

5.2.1 Modified Murthy Estimator – I:

This estimator has been developed by using the Brewer (1963a) selection procedure in the general Murthy estimator given as:

$$t_{symm} = \frac{1}{P(S)} \sum_{i=1}^{n} P(S|\,i)\, y_i \qquad (1.5.10)$$

Above estimator for a sample of size 2 is given as:

$$t_{symm} = \frac{P(S|\,i)y_i + P(S|\,j)y_j}{P(S)} \qquad (5.2.1)$$

Now, for Brewer selection procedure:

$$P(S|\,i) = \frac{p_j}{1-p_i}\ , \quad P(S|\,j) = \frac{p_i}{1-p_j}$$

and $$P(S) = \frac{2\,p_i\,p_j}{k}\left[\frac{1}{1-2\,p_i} + \frac{1}{1-2\,p_j}\right] = \frac{4\,p_i\,p_j\,(1-p_i-p_j)}{k\,(1-2\,p_i)(1-2\,p_j)}$$

where $$k = 1 + \sum_{i=1}^{N} \frac{p_i}{1-2\,p_i}$$

Substituting these values in equation (5.2.1):

$$t_{MM1} = \frac{y_i\,\dfrac{p_j}{1-p_i} + y_j\,\dfrac{p_i}{1-p_j}}{\dfrac{4\,p_i\,p_j\,(1-p_i-p_j)}{k\,(1-2\,p_i)(1-2\,p_j)}}$$

$$= \frac{p_i\,p_j\left[\dfrac{y_i}{p_i(1-p_i)} + \dfrac{y_j}{p_j(1-p_j)}\right]}{\dfrac{4\,p_i\,p_j\,(1-p_i-p_j)}{k\,(1-2\,p_i)(1-2\,p_j)}}$$

$$= \frac{\left[\dfrac{y_i}{p_i}(1-p_j) + \dfrac{y_j}{p_j}(1-p_i)\right]}{(1-p_i)(1-p_j)} \cdot \frac{k\,(1-2\,p_i)(1-2\,p_j)}{4\,(1-p_i-p_j)}$$

$$= \frac{k\,(1-2\,p_i)(1-2\,p_j)\left[\dfrac{y_i}{p_i}(1-p_j) + \dfrac{y_j}{p_j}(1-p_i)\right]}{4\,(1-p_i)(1-p_j)(1-p_i-p_j)} \qquad (5.2.2)$$

This estimator is a slight modification of the usual Murthy estimator given in equation (1.5.11).

5.2.2 Modified Murthy Estimator – II:

The modified Murthy estimator – II has been developed by using the Durbin (1967) selection procedure in the general Murthy estimator given in equation (5.2.1). Now, for Durbin (1967) selection procedure:

$$P(S|i)=\frac{p_j}{k}\left[\frac{1}{1-2p_i}+\frac{1}{1-2p_j}\right]=\frac{2p_j(1-p_i-p_j)}{k(1-2p_i)(1-2p_j)}$$

$$P(S|j)=\frac{p_i}{k}\left[\frac{1}{1-2p_i}+\frac{1}{1-2p_j}\right]=\frac{2p_i(1-p_i-p_j)}{k(1-2p_i)(1-2p_j)}$$

and $$P(S)=\frac{2p_i p_j}{k}\left[\frac{1}{1-2p_i}+\frac{1}{1-2p_j}\right]=\frac{4p_i p_j(1-p_i-p_j)}{k(1-2p_i)(1-2p_j)}$$

Now, substituting above values in equation (5.2.1) the estimator of population total for use with unequal probability sampling without replacement is:

$$t_{MM2}=\frac{\left[y_i\frac{2p_j(1-p_i-p_j)}{k(1-2p_i)(1-2p_j)}+y_j\frac{2p_i(1-p_i-p_j)}{k(1-2p_i)(1-2p_j)}\right]}{\frac{4p_i p_j(1-p_i-p_j)}{k(1-2p_i)(1-2p_j)}}$$

$$=\frac{k(1-2p_i)(1-2p_j)}{4p_i p_j(1-p_i-p_j)}\left[\frac{2y_i p_j(1-p_i-p_j)+2y_j p_i(1-p_i-p_j)}{k(1-2p_i)(1-2p_j)}\right]$$

$$=\frac{2y_i p_j(1-p_i-p_j)+2y_j p_i(1-p_i-p_j)}{4p_i p_j(1-p_i-p_j)}$$

$$=\frac{2p_i p_j(1-p_i-p_j)\left[\frac{y_i}{p_i}+\frac{y_j}{p_j}\right]}{4p_i p_j(1-p_i-p_j)}$$

$$=\frac{1}{2}\left[\frac{y_i}{p_i}+\frac{y_j}{p_j}\right] \qquad (5.2.3)$$

This estimator is same as used by Durbin (1967) for his rejective selection procedure.

5.2.3 Modified Murthy Estimator – III:

This estimator has been developed by using the new selection procedure – III in the general Murthy estimator given in equation (5.2.1). For new selection procedure – III, the conditional probabilities and the probability of sample are given as:

$$P(S|i)=\frac{p_j}{k}\left[\frac{1}{1-2p_i}+\frac{1}{1-2p_j}\right]=\frac{2p_j(1-p_i-p_j)}{k(1-2p_i)(1-2p_j)} \tag{5.2.4}$$

$$P(S|j)=\frac{p_i}{k}\left[\frac{1}{1-2p_i}+\frac{1}{1-2p_j}\right]=\frac{2p_i(1-p_i-p_j)}{k(1-2p_i)(1-2p_j)} \tag{5.2.5}$$

and $$P(S)=\frac{p_i p_j}{k^2}\left[\frac{1}{1-2p_i}+\frac{1}{1-2p_j}\right]\left[2+\frac{1}{1-2p_i}+\frac{1}{1-2p_j}\right]$$

$$=\frac{4p_i p_j(1-p_i-p_j)\{(1-p_i-p_j)+(1-2p_i)(1-2p_j)\}}{k^2(1-2p_i)^2(1-2p_j)^2} \tag{5.2.6}$$

Now, substituting the values from equation (5.2.4), (5.2.5) and (5.2.6) in equation (5.2.1) we have:

$$t_{MM3}=\frac{\left[y_i\frac{2p_j(1-p_i-p_j)}{k(1-2p_i)(1-2p_j)}+y_j\frac{2p_i(1-p_i-p_j)}{k(1-2p_i)(1-2p_j)}\right]}{\frac{4p_i p_j(1-p_i-p_j)\{(1-p_i-p_j)+(1-2p_i)(1-2p_j)\}}{k^2(1-2p_i)^2(1-2p_j)^2}}$$

$$=\left[\frac{2y_i p_j(1-p_i-p_j)+2y_j p_i(1-p_i-p_j)}{k(1-2p_i)(1-2p_j)}\right]$$

$$\cdot\frac{k^2(1-2p_i)^2(1-2p_j)^2}{4p_i p_j(1-p_i-p_j)[(1-p_i-p_j)+(1-2p_i)(1-2p_j)]}$$

$$=\frac{2k\,p_i p_j(1-p_i-p_j)(1-2p_i)(1-2p_j)\left[\frac{y_i}{p_i}+\frac{y_j}{p_j}\right]}{4p_i p_j(1-p_i-p_j)[(1-p_i-p_j)+(1-2p_i)(1-2p_j)]}$$

$$=\frac{k(1-2p_i)(1-2p_j)}{2[(1-p_i-p_j)+(1-2p_i)(1-2p_j)]}\left[\frac{y_i}{p_i}+\frac{y_j}{p_j}\right] \tag{5.2.7}$$

This estimator is a slight modification of the estimator given in equation (5.2.3).

5.2.4 Modified Raj Estimator:

This estimator has been developed following the idea of Raj (1956a). Raj (1956a) proposed a series of estimators based on the ordered samples. The estimators proposed by Raj (1956a) has a general shape as:

$$t_n = \sum_{i=1}^{n-1} y_i + \frac{y_n}{p_n}\left(1 - \sum_{i=1}^{n-1} p_i\right) \tag{1.7.4}$$

Raj (1956a) proposed that the average of above series of estimators can be used as an unbiased estimator of population total. Raj's estimator for n = 2 is given as:

$$t_2 = y_1 + \frac{y_2}{p_2}(1 - p_1) \tag{5.2.8}$$

Parallel to above estimator a modified estimator of population total is given as:

$$t_{RM} = y_j + \frac{y_i}{p_i}(1 - p_i)\left(b - \frac{p_j}{1 - p_j}\right) \tag{5.2.9}$$

where $b = \sum_{i=1}^{N} \frac{p_i}{1 - p_i}$

5.3 Unbiasedness of The Modified Estimators:

In this section the unbiasedness of modified estimators, developed in the previous section, has been proved. The unbiasedness has been proved by using directly the estimator and the probability of sample. The unbiasedness of all three estimators have been proved in the following subsections.

5.3.1 The Modified Murthy Estimator – I:

The unbiasedness of the modified Murthy estimator – I has been proved in this subsection as under:

Now $t_{MM1} = \dfrac{k(1-2p_i)(1-2p_j)\left[\dfrac{y_i}{p_i}(1-p_j) + \dfrac{y_j}{p_j}(1-p_i)\right]}{4(1-p_i)(1-p_j)(1-p_i-p_j)}$

also $P(S) = \dfrac{2p_i p_j}{k}\left[\dfrac{1}{1-2p_i} + \dfrac{1}{1-2p_j}\right] = \dfrac{4p_i p_j(1-p_i-p_j)}{k(1-2p_i)(1-2p_j)}$

Now

$$E\left(t_{MM1}\right)=\sum_{\substack{i=1\\j>i}}^{N}\sum_{j=1}^{N} t_{MM1}\, P(S)$$

$$=\frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N} t_{MM1}\, P(S)$$

$$=\frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\left[\frac{k\left(1-2P_i\right)\left(1-2P_j\right)\left[\frac{Y_i}{P_i}\left(1-P_j\right)+\frac{Y_j}{p_j}\left(1-P_i\right)\right]}{4\left(1-P_i\right)\left(1-P_j\right)\left(1-P_i-P_j\right)}\cdot\frac{4P_iP_j\left(1-P_i-P_j\right)}{k\left(1-2P_i\right)\left(1-2P_j\right)}\right]$$

$$=\frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\frac{P_iP_j\left[\frac{Y_i}{P_i}\left(1-P_j\right)+\frac{Y_j}{P_j}\left(1-P_i\right)\right]}{\left(1-P_i\right)\left(1-P_j\right)}$$

$$=\frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}P_iP_j\left[\frac{Y_i}{P_i\left(1-P_i\right)}+\frac{Y_j}{P_j\left(1-P_j\right)}\right]$$

$$=\frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}^{N}\left[\frac{Y_i}{1-P_i}P_j+\frac{Y_j}{1-P_j}P_i\right]$$

$$=\frac{1}{2}\left[\sum_{i=1}^{N}\frac{Y_i}{1-P_i}\sum_{\substack{j=1\\j\neq i}}^{N}P_j+\sum_{j=1}^{N}\frac{Y_j}{1-P_j}\sum_{\substack{i=1\\i\neq j}}^{N}P_i\right]$$

$$=\frac{1}{2}\left[\sum_{i=1}^{N}\frac{Y_i}{1-P_i}\left(1-P_i\right)+\sum_{j=1}^{N}\frac{Y_j}{1-P_j}\left(1-P_j\right)\right]$$

$$=\frac{1}{2}\left[\sum_{i=1}^{N}Y_i+\sum_{j=1}^{N}Y_j\right]$$

$$=\frac{1}{2}\left[Y+Y\right]$$

$$=\frac{1}{2}*2Y \;\Rightarrow\; E\left(t_{MM1}\right)=Y$$

From above result it can be seen that the modified Murthy estimator – I is unbiased for population total.

5.3.2 The Modified Murthy Estimator – II:

In this subsection the unbiasedness of modified Murthy estimator – II has been proved.

Now $t_{MM2} = \dfrac{1}{2}\left[\dfrac{y_i}{p_i} + \dfrac{y_j}{p_j}\right]$

also $P(S) = \dfrac{2\,p_i\,p_j}{k}\left[\dfrac{1}{1-2\,p_i} + \dfrac{1}{1-2\,p_j}\right] = \dfrac{4\,p_i\,p_j\,(1-p_i-p_j)}{k\,(1-2\,p_i)(1-2\,p_j)}$

So

$$E(t_{MM2}) = \sum_{i=1}^{N}\sum_{\substack{j=1\\ j>i}}^{N} t_{MM2}\,P(S)$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} t_{MM2}\,P(S)$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\left[\frac{1}{2}\left\{\frac{Y_i}{P_i}+\frac{Y_j}{P_j}\right\}\cdot\frac{4\,P_i\,P_j\,(1-P_i-P_j)}{k\,(1-2\,P_i)(1-2\,P_j)}\right]$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\left[\frac{1}{2}\left\{\frac{Y_i}{P_i}+\frac{Y_j}{P_j}\right\}\cdot\frac{2\,P_i\,P_j\,(2-2\,P_i-2\,P_j)}{k\,(1-2\,P_i)(1-2\,P_j)}\right]$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\left[\left\{\frac{Y_i}{P_i}+\frac{Y_i}{P_j}\right\}\cdot\frac{P_i\,P_j\,\{(1-2\,P_i)+(1-2\,P_j)\}}{k\,(1-2\,P_i)(1-2\,P_j)}\right]$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\left[\{Y_i\,P_i+Y_j\,P_i\}\cdot\frac{\{(1-2\,P_i)+(1-2\,P_j)\}}{k\,(1-2\,P_i)(1-2\,P_j)}\right]$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\left[\{Y_i\,P_j+Y_j\,P_i\}\cdot\left\{\frac{(1-2\,P_i)}{k\,(1-2\,P_i)(1-2\,P_j)}+\frac{(1-2\,P_j)}{k\,(1-2\,P_i)(1-2\,P_j)}\right\}\right]$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\left[\frac{Y_i\,P_j}{k\,(1-2\,P_i)}+\frac{Y_i\,P_j}{k\,(1-2\,P_j)}+\frac{Y_j\,P_i}{k\,(1-2\,P_i)}+\frac{Y_j\,P_i}{k\,(1-2\,P_j)}\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{N}\frac{Y_i}{k\left(1-2P_i\right)}\sum_{\substack{j=1\\ j\neq i}}^{N}P_j+\sum_{j=1}^{N}\frac{P_j}{k\left(1-2P_j\right)}\sum_{\substack{i=1\\ i\neq j}}^{N}Y_i\right.$$

$$\left.+\sum_{i=1}^{N}\frac{P_i}{k\left(1-2P_i\right)}\sum_{\substack{j=1\\ j\neq i}}^{N}Y_j+\sum_{j=1}^{N}\frac{Y_j}{k\left(1-2P_j\right)}\sum_{\substack{i=1\\ i\neq j}}^{N}P_i\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{N}\frac{Y_i}{k\left(1-2P_i\right)}\left(1-P_i\right)+\sum_{j=1}^{N}\frac{P_j}{k\left(1-2P_j\right)}\left(Y-Y_j\right)\right.$$

$$\left.+\sum_{i=1}^{N}\frac{P_i}{k\left(1-2P_i\right)}\left(Y-Y_i\right)+\sum_{j=1}^{N}\frac{Y_j}{k\left(1-2P_j\right)}\left(1-P_j\right)\right]$$

$$= \frac{1}{2k}\left[\sum_{i=1}^{N}\frac{Y_i}{\left(1-2P_i\right)}\left(1-P_i\right)+\sum_{j=1}^{N}\frac{P_j}{\left(1-2P_j\right)}\left(Y-Y_j\right)\right.$$

$$\left.+\sum_{i=1}^{N}\frac{P_i}{\left(1-2P_i\right)}\left(Y-Y_i\right)+\sum_{j=1}^{N}\frac{Y_j}{\left(1-2P_j\right)}\left(1-P_j\right)\right]$$

$$= \frac{1}{2k}\left[\sum_{i=1}^{N}\frac{Y_i}{\left(1-2P_i\right)}-\sum_{i=1}^{N}\frac{Y_i P_i}{\left(1-2P_i\right)}+Y\sum_{j=1}^{N}\frac{P_j}{\left(1-2P_j\right)}-\sum_{j=1}^{N}\frac{Y_j P_j}{\left(1-2P_j\right)}\right.$$

$$\left.+Y\sum_{i=1}^{N}\frac{P_i}{\left(1-2P_i\right)}-\sum_{i=1}^{N}\frac{Y_i P_i}{\left(1-2P_i\right)}+\sum_{j=1}^{N}\frac{Y_j}{\left(1-2P_j\right)}-\sum_{j=1}^{N}\frac{Y_j P_j}{\left(1-2P_j\right)}\right]$$

$$= \frac{1}{2k}\left[2Y\sum_{i=1}^{N}\frac{P_i}{\left(1-2P_i\right)}+2\sum_{i=1}^{N}\frac{Y_i}{\left(1-2P_i\right)}-4\sum_{i=1}^{N}\frac{Y_i P_i}{\left(1-2P_i\right)}\right]$$

$$= \frac{1}{2k}\left[2Y\sum_{i=1}^{N}\frac{P_i}{\left(1-2P_i\right)}+2\sum_{i=1}^{N}\frac{Y_i}{\left(1-2P_i\right)}\left(1-2P_i\right)\right]$$

$$= \frac{1}{2k}\left[2Y\sum_{i=1}^{N}\frac{P_i}{\left(1-2P_i\right)}+2\sum_{i=1}^{N}Y_i\right]$$

$$= \frac{1}{2k}\left[2Y\sum_{i=1}^{N}\frac{P_i}{\left(1-2P_i\right)}+2Y\right]$$

$$= \frac{1}{2k}\left[2Y\left\{1+\sum_{i=1}^{N}\frac{P_i}{1-2P_i}\right\}\right]$$

$$= \frac{1}{2k}[2kY] \Rightarrow E(t_{MM2}) = Y$$

which is unbiased. It is worth to quote here that the same estimator, when used by Durbin (1967) with his rejective selection procedure, turned out to be a biased estimator of population total Y.

5.3.3 The Modified Murthy Estimator – III:

In this subsection the unbiasedness of modified Murthy estimator – III has been proved. From subsection 5.2.3:

$$t_{MM3} = \frac{k(1-2p_i)(1-2p_j)}{2[(1-p_i-p_j)+(1-2p_i)(1-2p_j)]}\left[\frac{y_i}{p_i}+\frac{y_j}{p_j}\right]$$

and $$P(S) = \frac{4p_i p_j(1-p_i-p_j)\{(1-p_i-p_j)+(1-2p_i)(1-2p_j)\}}{k^2(1-2p_i)^2(1-2p_j)^2}$$

Now, the unbiasedness of t_{MM3} has been proved as under:

$$E(t_{MM3}) = \sum_{i=1}^{N}\sum_{\substack{j=1\\ j>i}}^{N} t_{MM3}\, P(S)$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N} t_{MM3}\, P(S)$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\left[\left\{\frac{k(1-2P_i)(1-2P_j)}{2[(1-P_i-P_j)+(1-2P_i)(1-2P_j)]}\left(\frac{Y_i}{P_i}+\frac{Y_j}{P_j}\right)\right\}\right.$$

$$\left.\cdot\left\{\frac{4P_iP_j(1-P_i-P_j)\{(1-P_i-P_j)+(1-2P_i)(1-2P_j)\}}{k^2(1-2P_i)^2(1-2P_j)^2}\right\}\right]$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\left[\left\{\frac{Y_i}{P_i}+\frac{Y_j}{P_j}\right\}\cdot\left\{\frac{2P_iP_j(1-P_i-P_j)}{k(1-2P_i)(1-2P_j)}\right\}\right]$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}\left[\left\{\frac{Y_i}{P_i}+\frac{Y_j}{P_j}\right\}\cdot\frac{P_i P_j\left(2-2P_i-2P_j\right)}{k\left(1-2P_i\right)\left(1-2P_j\right)}\right]$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}\left[\left\{\frac{Y_i P_j+Y_j P_i}{P_i P_j}\right\}\cdot\frac{P_i P_j\left\{\left(1-2P_i\right)+\left(1-2P_j\right)\right\}}{k\left(1-2P_i\right)\left(1-2P_j\right)}\right]$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}\left[\left\{Y_i P_j+Y_j P_i\right\}\cdot\left\{\frac{\left(1-2P_i\right)}{k\left(1-2P_i\right)\left(1-2P_j\right)}+\frac{\left(1-2P_j\right)}{k\left(1-2P_i\right)\left(1-2P_j\right)}\right\}\right]$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}\left[\left\{Y_i P_j+Y_j P_i\right\}\cdot\left\{\frac{1}{k\left(1-2P_j\right)}+\frac{1}{k\left(1-2P_i\right)}\right\}\right]$$

$$= \frac{1}{2}\sum_{i=1}^{N}\sum_{\substack{j=1\\j\neq i}}\left[\frac{Y_i P_j}{k\left(1-2P_i\right)}+\frac{Y_i P_j}{k\left(1-2P_j\right)}+\frac{Y_j P_i}{k\left(1-2P_i\right)}+\frac{Y_j P_i}{k\left(1-2P_j\right)}\right]$$

$$= \frac{1}{2}\left[\sum_{i=1}^{N}\frac{Y_i}{k\left(1-2P_i\right)}\sum_{\substack{j=1\\j\neq i}}^{N}P_j+\sum_{j=1}^{N}\frac{P_j}{k\left(1-2P_j\right)}\sum_{\substack{i=1\\i\neq j}}^{N}Y_i\right.$$

$$\left.+\sum_{i=1}^{N}\frac{P_i}{k\left(1-2P_i\right)}\sum_{\substack{j=1\\j\neq i}}^{N}Y_j+\sum_{j=1}^{N}\frac{Y_j}{k\left(1-2P_j\right)}\sum_{\substack{i=1\\i\neq j}}^{N}P_i\right]$$

$$= \frac{1}{2k}\left[\sum_{i=1}^{N}\frac{Y_i}{\left(1-2P_i\right)}\left(1-P_i\right)+\sum_{j=1}^{N}\frac{P_j}{\left(1-2P_j\right)}\left(Y-Y_j\right)\right.$$

$$\left.+\sum_{i=1}^{N}\frac{P_i}{\left(1-2P_i\right)}\left(Y-Y_i\right)+\sum_{j=1}^{N}\frac{Y_j}{\left(1-2P_j\right)}\left(1-P_j\right)\right]$$

$$= \frac{1}{2k}\left[\sum_{i=1}^{N}\frac{Y_i}{\left(1-2P_i\right)}-\sum_{i=1}^{N}\frac{Y_i P_i}{\left(1-2P_i\right)}+Y\sum_{j=1}^{N}\frac{P_j}{\left(1-2P_j\right)}-\sum_{j=1}^{N}\frac{Y_j P_j}{\left(1-2P_j\right)}\right.$$

$$\left.+Y\sum_{i=1}^{N}\frac{P_i}{\left(1-2P_i\right)}-\sum_{i=1}^{N}\frac{Y_i P_i}{\left(1-2P_i\right)}+\sum_{j=1}^{N}\frac{Y_j}{\left(1-2P_j\right)}-\sum_{j=1}^{N}\frac{Y_j P_j}{\left(1-2P_j\right)}\right]$$

$$= \frac{1}{2k}\left[2Y\sum_{i=1}^{N}\frac{P_i}{\left(1-2P_i\right)}+2\sum_{i=1}^{N}\frac{Y_i}{\left(1-2P_i\right)}-4\sum_{i=1}^{N}\frac{Y_i P_i}{\left(1-2P_i\right)}\right]$$

$$= \frac{1}{2k}\left[2Y\sum_{i=1}^{N}\frac{P_i}{(1-2P_i)}+2\sum_{i=1}^{N}\frac{Y_i}{(1-2P_i)}(1-2P_i)\right]$$

$$= \frac{1}{2k}\left[2Y\sum_{i=1}^{N}\frac{P_i}{(1-2P_i)}+2Y\right]$$

$$= \frac{1}{2k}\left[2Y\left\{1+\sum_{i=1}^{N}\frac{P_i}{1-2P_i}\right\}\right]$$

$$= \frac{1}{2k}[2kY] \Rightarrow E(t_{MM3}) = Y$$

which proves the unbiasedness of modified Murthy estimator – III.

5.3.4 The Modified Raj Estimator:

In this sub – section the unbiasedness of the modified Raj estimator has been proved. To prove that unbiasedness consider the estimator given in equation (5.11) as:

$$t_{RM} = y_j + \frac{y_i}{p_i}(1-p_i)\left(b-\frac{p_j}{1-p_j}\right)$$

Since above is an ordered estimator, therefore for this estimator:

$$P(S) = \frac{p_i\, p_j}{1-p_j}$$

Now to prove unbiasedness of t_{RM} we proceed as under:

$$E(t_{RM}) = \sum_{\substack{i=1\\}}^{N}\sum_{\substack{j=1\\ j\neq i}} t_{RM}\, P(S)$$

$$= \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}\left[Y_j + \frac{Y_i}{P_i}(1-P_i)\left(b-\frac{P_j}{1-P_j}\right)\right]\frac{P_i P_j}{1-P_j}$$

$$= \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}\left[Y_j + Y_i\left(\frac{1-P_i}{P_i}\right)\left(b-\frac{P_j}{1-P_j}\right)\right]\frac{P_i P_j}{1-P_j}$$

$$= \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}} Y_j\frac{P_i P_j}{1-P_i} + \sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}\left[Y_i\left(\frac{1-P_i}{P_i}\right)\left(b-\frac{P_j}{1-P_j}\right)\right]\frac{P_i P_j}{1-P_j}$$

$$= \sum_{\substack{i=1 \\ j \neq i}}^{N} \sum_{j=1} Y_j \frac{P_i P_j}{1-P_i} + \sum_{\substack{i=1 \\ j \neq i}}^{N} \sum_{j=1} Y_i \left(b - \frac{P_j}{1-P_j} \right) P_j$$

$$= \sum_{\substack{i=1 \\ j \neq i}}^{N} \sum_{j=1} Y_j \frac{P_i P_j}{1-P_i} + \sum_{\substack{i=1 \\ j \neq i}}^{N} \sum_{j=1} b Y_i P_j - \sum_{\substack{i=1 \\ j \neq i}}^{N} \sum_{j=1} Y_i \frac{P_j^2}{1-P_j}$$

$$= \sum_{j=1}^{N} Y_j P_j \sum_{\substack{i=1 \\ i \neq j}}^{N} \frac{P_i}{1-P_i} + \sum_{i=1}^{N} b Y_i \sum_{\substack{j=1 \\ j \neq i}}^{N} P_j - \sum_{i=1}^{N} Y_i \sum_{\substack{j=1 \\ j \neq i}}^{N} \frac{P_j^2}{1-P_j}$$

$$= \sum_{j=1}^{N} Y_j P_j \left[\sum_{i=1}^{N} \frac{P_i}{1-P_i} - \frac{P_j}{1-P_j} \right] + b \sum_{i=1}^{N} Y_i (1-P_i) - \sum_{i=1}^{N} Y_i \left[\sum_{j=1}^{N} \frac{P_j^2}{1-P_j} - \frac{P_i^2}{1-P_i} \right]$$

$$= \sum_{j=1}^{N} Y_j P_j \left[b - \frac{P_j}{1-P_j} \right] + b \sum_{i=1}^{N} Y_i (1-P_i) - \sum_{i=1}^{N} Y_i \left[\sum_{j=1}^{N} \frac{P_j^2}{1-P_j} - \frac{P_i^2}{1-P_i} \right]$$

$$= b \sum_{i=1}^{N} Y_i - \sum_{i=1}^{N} Y_i \sum_{i=1}^{N} \frac{P_i^2}{1-P_i}$$

$$= \sum_{i=1}^{N} Y_i \sum_{i=1}^{N} \frac{P_i}{1-P_i} - \sum_{i=1}^{N} Y_i \sum_{i=1}^{N} \frac{P_i^2}{1-P_i}$$

$$= \sum_{i=1}^{N} Y_i \left[\sum_{i=1}^{N} \frac{P_i}{1-P_i} - \sum_{i=1}^{N} \frac{P_i^2}{1-P_i} \right]$$

$$= \sum_{i=1}^{N} Y_i \left[\sum_{i=1}^{N} \frac{P_i}{1-P_i} (1-P_i) \right]$$

$$= \sum_{i=1}^{N} Y_i \sum_{i=1}^{N} P_i$$

$$= \sum_{i=1}^{N} Y_i \quad \Rightarrow \quad E(t_{RM}) = Y$$