CHAPTER 2

ON ANALYTIC FUNCTIONS WITH GENERALIZED BOUNDED MOCANU VARIATION

2.1 Introduction

It is well known [28] that a number of important classes of univalent functions (e.g. convex, starlike) are related through their derivatives by functions with positive real part. These functions play an important part in problem from signal theory, in moment problems and in constructing quadrature formulas, see Ronning [97] and the references cited therein for some recent applications. In this chapter, we introduce and consider some new classes of functions by replacing functions with positive real part by certain weighted differences of such functions.

In section 2.2, we define a new class \( B_k(\alpha, \beta) \) for real \( \alpha, \frac{-1}{2} \leq \beta < 1, \ k \geq 2 \) and \( z \in E \).

For different choices of parameters \( k, \alpha \) and \( \beta \) we presents its relationships with the previously known classes. It is known [113] that \( B_z(\alpha, \beta) \) is a subclass of Bazilevic functions defined in [7], and consists entirely of univalent functions. We study the relationship between the classes \( B_k(\alpha, \beta) \) and \( R_k \), where \( R_k \) denotes the class of bounded radius rotation, see [36]. In this section we focus on the inclusion results between the classes \( B_k(\alpha, \beta) \) and \( B_{h_l}(\alpha, \beta) \). We also eastablish the criterion of univlance for the class \( B_k(\alpha, \beta) \) by restricting the parameter \( k \leq \frac{2(\alpha + 2\alpha\beta - \beta + 1)}{(1 - \beta)}, \ \alpha \neq 0 \). As a special case, we deduce that \( f \in B_z(\alpha, \beta) \) is univalent in \( E \) for \( \alpha > 0, \ \frac{-1}{2} \leq \beta < 1 \).
In section 2.3, we derive arc lengths for the class \( B_k(\alpha, \beta) \). In addition, we characterize the growth of the coefficient for the class \( B_k(\alpha, \beta) \) by specializing the parameter \( \alpha > 0 \).

In section 2.4, we have shown that the class \( B_k(0, \beta) \) preserves the Barnardi integral operator. Further we have discussed in details the sharp bounds of the class \( B_2(\alpha, \beta) \) for \( \alpha \neq 0 \) and \( 0 < \beta < 1 \), by using the Gauss hypergeometric function. Finally, in this section, for different parameters of interest we obtained the coefficient bounds by assuming \( a_2 \) to be real. We also derive a covering and distortion theorems for the classes \( B_2(\alpha, \beta) \) and \( B_k(1, \beta) \) by restricting the parameters \( k, \alpha \) and \( \beta \). The contents of this chapter have been published in the journal of Applied Mathematics and Computations, see [76].

We now define the following class.

\[ 2.2 \quad \text{The class } B_k(\alpha, \beta) \]

**Definition 2.2.1.**

Let \( f \in \mathcal{A} \) with \( \frac{f(z)f'(z)}{z} \neq 0 \) in \( E \), and let

\[
J(\alpha, \beta, k, f(z)) = \left[ (1-\alpha) \frac{zf'(z)}{f(z)} + \frac{\alpha}{1-\beta}\left[1-\beta + \frac{zf''(z)}{f'(z)}\right]\right],
\]

for real \( \alpha \) and \( -\frac{1}{2} \leq \beta < 1 \).

Then \( f \in B_k(\alpha, \beta) \) if and only if \( J(\alpha, \beta, k, f(z)) \in P_k \) for \( z \in E, \ k \geq 2 \).

For any real \( \alpha \) and \( -\frac{1}{2} \leq \beta < 1 \), we note that the identity function belongs to \( B_k(\alpha, \beta) \) so that \( B_k(\alpha, \beta) \) is not empty.

**Special cases.**

(i). For \( k = 2, \ 0 \leq \alpha < 1 \), \( B_2(\alpha, 0) \) is a subclass of \( \mathcal{A} \) introduced by Mocanu [60]. Also, see [57, 75].
(ii). It is well known [113] that $B_2(\alpha, \beta)$ is a subclass of Bazilevic functions defined in [7]. Hence $B_2(\alpha, \beta)$ consists entirely of univalent functions.

(iii). $B_k(1,0) = V_k$ is the well known class of functions of bounded boundary rotation.

(iv). $B_k(0,\beta) = R_k$, where $R_k$ denotes the class of bounded radius rotation, see [36].

(v). $B_2(1,0) = C, \quad B_2(0,\beta) = S^*$, where $C$ and $S^*$ are respectively the classes of convex and starlike univalent functions in $E$.

### 2.2.1 Relation between the classes $B_k(\alpha, \beta)$ and $R_k$

#### Theorem 2.2.2.

Let $f \in B_k(\alpha, \beta), \quad \alpha \neq 0$, if and only if there is a function $g \in B_k(0,\beta) = R_k$ such that

$$f(z) = \left[ mtm^{-1}\left(\frac{g(t)}{t}\right)^{\frac{1-\alpha}{\beta}} dt \right]^{\frac{1}{m}} = z + ...,$$

where $m = 1 + \frac{(1-%\alpha)(1-%\beta)}{\alpha}$.

**Proof.**

A simple computation yields

$$(1-%\alpha)\frac{zf'(z)}{f(z)} + \frac{\alpha}{1-%\beta}\left(1-%\beta + \frac{zf^*(z)}{f'(z)}\right) = \frac{zg'(z)}{g(z)}.$$

If the right hand side belongs to $R_k$, so does the left hand side and conversely.

#### Theorem 2.2.3.

Let $f \in B_k(\alpha, \beta)$. Then the function

$$g(z) = z\left( \frac{f(z)}{z} \right)^{1-%\alpha} \left( f'(z) \right)^{\frac{\alpha}{1-%\beta}} \tag{2.2.1}$$
belongs to \( R_k \) for all \( z \in E \).

**Proof.**

Differentiating (2.2.1) logarithmically, we have

\[
\frac{zg'(z)}{g(z)} = (1-\alpha) \frac{zf''(z)}{f(z)} + \alpha + \frac{\alpha}{(1-\beta)} \left( \frac{zf''(z)}{f'(z)} \right)
\]

and result follows immediately since \( f \in B_k(\alpha, \beta) \).

**Theorem 2.2.4.**

\( B_k(\alpha, \beta) \subset R_k \), for \( \alpha > 0 \), \( 0 \leq \beta < 1 \).

**Proof.**

Let

\[
\frac{zf'(z)}{f(z)} = p(z),
\]

\( p(z) \) is analytic in \( E \) with \( p(0) = 1 \). Now

\[
\frac{1}{1-\beta} \left[ (1-\alpha)(1-\beta) \frac{zf''(z)}{f(z)} + \alpha \left( 1-\beta + \frac{zf''(z)}{f'(z)} \right) \right] = \frac{\alpha}{1-\beta} \left[ m \left\{ p(z) + \frac{1}{m} \frac{zp'(z)}{p(z)} \right\} - \beta \right] \in P_k, \quad z \in E.
\]

This implies \( \left\{ p(z) + \frac{1}{m} \frac{zp'(z)}{p(z)} \right\} \in P_k \) and, by Lemma 1.6.8, it follows that

\( p \in P_k, \quad z \in E \). This proves that \( f \in R_k \) in \( E \).

**2.2.2 Inclusion results between the classes \( B_k(\alpha, \beta) \) and \( B_{k_i}(\alpha_i, \beta) \)**

**Theorem 2.2.5.**

(i) \( B_k(\alpha, \beta) \subset B_{k_i}(\alpha_i, \beta) \), \( 0 < \alpha \leq \alpha_i \) and \( k_i = k \left( \frac{2\alpha_i - \alpha}{\alpha} \right) \).
(ii) \( B_k(\alpha, \beta) \subset B_{k_1}(\alpha_1, \beta), \quad 0 < \alpha_i \leq \alpha. \)

**Proof.**

(i) Let \( f \in B_k(\alpha, \beta). \) Then

\[
(1 - \alpha_i)(1 - \beta) \frac{zf'(z)}{f(z)} + \alpha_i \left( 1 - \beta + \frac{zf''(z)}{f'(z)} \right)
\]

\[
= \frac{\alpha_i}{\alpha} \left[ (1 - \alpha)(1 - \beta) \frac{zf'(z)}{f(z)} + \alpha \left( 1 - \beta + \frac{zf''(z)}{f'(z)} \right) \right] \frac{(1 - \beta)(\alpha_i - \alpha) \frac{zf'(z)}{f(z)}}{\alpha} \]

\[
= (1 - \beta) \left[ \frac{\alpha_i}{\alpha} h_1(z) - \frac{\alpha_i - \alpha}{\alpha} h_2(z) \right], \quad h_1, h_2 \in P_k, \quad (2.2.2)
\]

by using Definition 2.2.1 and Theorem 2.2.4.

From (2.2.2), it follows that

\[
\int_0^{2\pi} \left| \text{Re} J(\alpha_i, \beta, f(z)) \right| d\theta \leq \left( \frac{\alpha_i}{\alpha} + \frac{\alpha_i - \alpha}{\alpha} \right) k\pi
\]

\[
= \left( \frac{2\alpha_i - \alpha}{\alpha} \right) k\pi.
\]

This completes the proof of (i).

(ii) Let \( f \in B_k(\alpha, \beta). \) Then

\[
(1 - \alpha_i) \frac{zf'(z)}{f(z)} + \frac{\alpha_i}{(1 - \beta)} \left( 1 - \beta + \frac{zf''(z)}{f'(z)} \right)
\]

\[
= (1 - \frac{\alpha_i}{\alpha}) \frac{zf'(z)}{f(z)} + \frac{\alpha_i}{\alpha} \left[ (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha + \frac{\alpha}{(1 - \beta)} \left( 1 - \beta + \frac{zf''(z)}{f'(z)} \right) \right]
\]

\[
= (1 - \frac{\alpha_i}{\alpha}) H_1(z) + \frac{\alpha_i}{\alpha} H_2(z), \quad H_1, H_2 \in P_k, \quad z \in E,
\]

since \( P_k \) is a convex set, see [70]. Therefore \( f \in B_k(\alpha, \beta) \) for \( z \in E. \)
2.2.3 The condition of univalency for the class $B_k(\alpha, \beta)$

**Theorem 2.2.6.**

Let $f \in B_k(\alpha, \beta)$. Then $f$ is univalent in $E$ for

$$k \leq \frac{2(\alpha + 2\alpha\beta - \beta + 1)}{(1 - \beta)}, \quad \alpha \neq 0.$$ 

**Proof.**

Since $f \in B_k(\alpha, \beta)$ we can write, for $z = re^{i\theta}$, $0 \leq r < 1$, $0 \leq \theta_1 < \theta_2 \leq 2\pi$,

$$\int_{\theta_1}^{\theta_2} \text{Re}\left\{ \frac{(1 - \alpha) zf''(z)}{\alpha f(z)} + \frac{1}{1 - \beta}\left(1 + \frac{zf''(z)}{f'(z)}\right) \right\} d\theta > -\left(\frac{k}{2} - 1\right)\frac{\pi}{\alpha} + \frac{2\beta}{1 - \beta}\pi.$$ 

This implies, by using Lemma 1.6.6, that $f$ is univalent in $E$ if

$$k \leq \frac{2(\alpha + 2\alpha\beta - \beta + 1)}{(1 - \beta)}.$$ 

As a special case, we deduce that $f \in B_k(\alpha, \beta)$ is univalent in

$E$ for $\alpha > 0$, $-\frac{1}{2} \leq \beta < 1$.

2.2 Arc length problems and growth of the coefficient for the class $B_k(\alpha, \beta)$

**Theorem 2.3.1.**

Let $f \in B_k(\alpha, \beta)$, $\alpha > 0$ and $L_r(f)$ denote the length of the curve

$$C, \quad C = f(re^{i\theta}), 0 \leq \theta \leq 2\pi \quad \text{and} \quad M(r) = \max_{0 \leq \theta \leq 2\pi} \left|f(re^{i\theta})\right|.$$ 

Then, for $0 < r < 1$. 

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(i) \( L_r(f) \leq \frac{M(r)}{(1-\beta)} \{k(1-\beta) + 2\beta\} \pi, \quad \alpha \geq 2. \)

(ii) \( L_r(f) \leq \frac{M(r)}{\alpha(1-\beta)} \{k(1-\beta)(2-\alpha) + 2\alpha\beta\} \pi, \quad 0 < \alpha < 2. \)

**Proof.**

(i) With \( z = re^{i\theta} \),

\[
L_r(f) = \int_0^{2\pi} |zf'(z)| d\theta = \int_0^{2\pi} zf'(z)e^{-i\arg(zf'(z))} d\theta.
\]

Integration by parts gives us

\[
L_r(f) = \int_0^{2\pi} f(z)e^{-i\arg(zf'(z))} Re \left\{ \frac{(zf'(z))'}{f'(z)} \right\} d\theta
\]

\[
\leq \frac{M(r)}{\alpha} \int_0^{2\pi} \left| Re J(\alpha, \beta, f(z)) + (\alpha - 1) \frac{zf'(z)}{f(z)} + \frac{\alpha\beta}{1 - \beta} \right| d\theta
\]

\[
\leq \frac{M(r)}{\alpha} \left[ k\pi + (\alpha - 1)k\pi + \frac{2\alpha\beta}{1 - \beta} \right]
\]

\[
= \pi M(r) \left[ k + \frac{2\beta}{1 - \beta} \right].
\]

Proof of (ii) lies on similar lines.

**Theorem 2.3.2.**

Let \( f \), given by (1.2.1), belongs to \( B_k(\alpha, \beta) \) for \( \alpha > 0 \). Then, for \( n \geq 2 \).

\[
n a_n = O(1)M\left(\frac{n-1}{n}\right),
\]

where \( O(1) \) is a constant depending on \( \alpha, \beta \) and \( k \) only.
Proof.

Since
\[ na_n = \frac{1}{2\pi r^n} \int_0^{2\pi} zf'(z)e^{-inz}d\theta, \quad z = re^{i\theta}, \]
we have
\[ n|a_n| \leq \frac{1}{2\pi r^n} \int_0^{2\pi} |zf'(z)|d\theta = \frac{1}{2\pi r^n} L_r(f). \]

Now using Theorem 2.3.1 with \( r = 1 - \frac{1}{n} \), we prove the result.

### 2.4 Properties of the special classes of the class \( B_k(\alpha, \beta) \)

#### 2.4.1 Integral preserving property of the class \( B_k(0, \beta) \)

**Theorem 2.4.1.**

Let, for \( \gamma > 0 \), \( F \) be defined as
\[ F(z) = \frac{\gamma + 1}{z\gamma} \int_0^1 r^{-1} f(t)dt, \quad (2.4.1) \]
and let \( f \in B_k(0, \beta) \). Then \( F \in B_k(0, \beta) \) in \( E \).

**Proof.**

Set
\[ \frac{zf'(z)}{F(z)} = H(z) = \left( \frac{k}{4} + \frac{1}{2} \right) h_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) h_2(z), \]
where \( H(z) \) is analytic in \( E \) with \( H(0) = 1 \). Then from (2.4.1), we have
\[ \frac{zf'(z)}{f(z)} = \left( H(z) + \frac{zH'(z)}{H(z) + \gamma} \right) \in P_k, \quad z \in E. \]

Now
\[(H \ast \phi_i)(z) = H(z) + \frac{zH'(z)}{H(z) + \gamma}\]

\[
= \left(\frac{k + 1}{4} + \frac{1}{2}\right)\left\{h_i(z) + \frac{zh_i'(z)}{h_i(z) + \gamma}\right\} - \left(\frac{k - 1}{4} + \frac{1}{2}\right)\left\{h_i(z) + \frac{zh_i'(z)}{h_i(z) + \gamma}\right\},
\]

where

\[
\phi_i(z) = \frac{1}{\gamma + 1}\left(\frac{z}{1 - z} + \frac{\gamma z}{1 + \gamma (1 - z)^2}\right).
\]

It follows that \(\left\{h_i(z) + \frac{zh_i'(z)}{h_i(z) + \gamma}\right\} \in P\), \(z \in E\), \(i = 1, 2\). We want to show that \(h_i \in P\) in \(E\) which maps \(H \in P_k\) in \(E\). We proceed by forming the functional \(\Psi(u,v)\) with \(u = h_i(z), \ v = zh_i'(z)\).

Thus

\[
\Psi(u,v) = u + \frac{v}{u + \gamma}.
\]

The first two conditions of Lemma 1.6.7 are clearly satisfied. We verify the condition (iii) as follows.

\[
\text{Re} \\Psi(iu, v) = \frac{v \gamma}{\gamma^2 + u^2},
\]

and by putting

\[
v_i \leq \frac{-(1 + u_i^2)}{2},
\]

we have

\[
\text{Re} \\Psi(iu, v) \leq \frac{-\gamma(1 + u_i^2)}{2\left(\gamma^2 + u_i^2\right)} \leq 0.
\]

This proves \(h_i \in P\), \(i = 1, 2\) and the proof is complete.
2.4.2 Some Sharp bounds and distortion theorem for the

class $B_2(\alpha, \beta)$

In the following, we study the distortion theorems for the class $B_k(\alpha, \beta)$ with some
restrictions on $k, \alpha$ and $\beta$. We shall use the hypergeometric functions.

Let $G(a, b, c; z)$ be the analytic functions for $z$ in $E$ as defined by (1.6.7) that is

$$G(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 v^{a-1}(1-v)^{c-a-1}(1-zv)^{-b} dv,$$

where $\text{Re}\{a\} > 0$ and $\text{Re}\{c-a\} > 0$. In addition, we define the functions:

$$M(\alpha, \beta, r) = r \left[ G \left( \frac{2(1-\beta)}{\alpha}, m, m+1, r \right) \right]^{\frac{1}{m}}, \quad (2.4.2)$$

and

$$f_\theta(\alpha, \beta, z) = \left[ \int_0^z t^{m-1}(1-e^{-\theta t})^{-\frac{2(1-\beta)}{\alpha}} dt \right]^{\frac{1}{m}}, \quad (2.4.3)$$

where

$$m = 1 + \frac{(1-\alpha)(1-\beta)}{\alpha}, \quad 0 \leq \theta \leq 2\pi.$$

We now prove the following.

**Theorem 2.4.2.**

Let $f \in B_2(\alpha, \beta), \quad \alpha \neq 0, \quad 0 < \beta < 1$, and $|z| = r \quad (0 < r < 1).$ Then

(i) $M(\alpha, \beta, -r) \leq |f(z)| \leq M(\alpha, \beta, r), \quad \text{for} \quad \alpha > 0.$

(ii) $M(\alpha, \beta, r) \leq |f(z)| \leq M(\alpha, \beta, -r), \quad \text{for} \quad \alpha < 0.$

This result is sharp and equality occurs, for the function $f_\theta(\alpha, \beta, z)$, defined by (2.4.3),
with suitably chosen $\theta$. 
Proof.

We consider $\alpha > 0$ and the case $\alpha < 0$ follows on similar lines.

From Theorem 3.3.1, for $f \in B_2(\alpha, \beta)$ if and only if, there exists a $g \in R_2 = S^*$ such that

$$f(z) = \left[ m \int_0^r t^{m-1} \left( \frac{g(t)}{t} \right)^{\frac{\alpha-1}{\alpha}} \, dt \right] \frac{1}{m^\frac{1}{m}},$$

where

$$m = 1 + \frac{(1-\alpha)(1-\beta)}{\alpha}, \quad (\alpha > 0).$$

We may take $z = r$, for the general case can be reduced to this by considering the function $\frac{f(\eta z)}{z}$ with suitably chosen $\eta$ such that $|\eta| = 1$. Taking $z = r$ and integrating along the positive real axis, we obtain from (2.4.4), for $t = \rho e^{i\theta}$,

$$f(r) = \left[ m r^{(m-1)\theta} \int_0^r \rho^{m-1} \left( \frac{g(\rho)}{\rho} \right)^{\frac{\alpha-1}{\alpha}} \, d\rho \right] \frac{1}{m^\frac{1}{m}}.$$  

Since $g$ is starlike, we have

$$\frac{\rho}{(1+\rho)^2} \leq |g(\rho)| \leq \frac{\rho}{(1-\rho)^2}.$$  

Using (2.4.6) in (2.4.5), we obtain

$$|f(r)| \leq m \int_0^r \rho^{m-1} (1-\rho)^{\frac{-2(1-\beta)}{\alpha}} \, d\rho$$

$$= mr \int_0^1 u^{m-1} (1-ru)^{\frac{-2(1-\beta)}{\alpha}} \, du, \quad (\rho = ru).$$

Therefore,

$$|f(r)| \leq M(\alpha, \beta, r), \quad \text{for} \quad \alpha > 0.$$
To prove the left hand inequality, we consider the straight line $\Gamma$ joining 0 to $f(z) = Re^{i\phi}$. $\Gamma$ is the image of a Jordan arc $\gamma$ in $E$ containing 0 and $z = re^{i\phi}$. Suppose now $z_0$ is a point on the circumference $|z| = r$ such that

$$|f(z_0)| = \min_{0 \leq \theta \leq 2\pi} |f(re^{i\theta})|.$$

We consider $\alpha > 0$. From (2.4.5) and (2.4.6), we obtain

$$|f(z_0)|^m \geq m \int_0^r \rho^{m-1} (1+\rho)^{\alpha \frac{-2(1-\beta)}{\alpha}} d\rho$$

$$= m r^m \int_0^1 u^{m-1} (1+ru)^{\alpha \frac{-2(1-\beta)}{\alpha}} du,$$

and this implies $|f(z)| \geq M(\alpha, \beta, -r), \quad \alpha > 0$.

Using Definition 2.2.1, relation (2.4.4) and Theorem 2.4.2, we have:

**Theorem 2.4.3.**

Let $f \in B_2(\alpha, \beta), \quad \alpha > 0$, Then, for $|z| = r \quad (0 < r < 1)$, we have

$$\frac{r + |\alpha - 1|(1+r^2)M(\alpha, 0, -r)}{\alpha r(1+r)^2} \leq |f'(z)| \leq \frac{r + |\alpha - 1|(1-r^2)M(\alpha, 0, r)}{\alpha r(1-r)^2}.$$

This result is sharp.

**2.4.3 Coefficient bounds and covering theorem for the class $B_2(\alpha, \beta)$**

**Theorem 2.4.4.**

Let $f \in B_2(\alpha, \beta) \quad 0 < \beta < 1, \quad \alpha \neq 0$, and be given by (3.1.1).
Then
\[|a_2| \leq \frac{2(1-\beta)}{[(1-\alpha)(1-\beta) + 2\alpha]}.
\]

For the proof we can assume \(a_2\) to be real and make use of Theorem 3.3.9 (i) for \(\alpha > 0\). The case \(\alpha < 0\) is similar. We can write
\[
M(\alpha, \beta, r) = r + \frac{2(1-\beta)}{(1-\alpha)(1-\beta) + 2\alpha} r^2 + O(r^3),
\]
and
\[
|f(r)| = r + a_2 r^2 + O(r^3).
\]
Therefore, we have
\[
a_2 \leq \frac{2(1-\beta)}{(1-\alpha)(1-\beta) + 2\alpha}, \quad (\alpha > 0).
\]

**Theorem 2.4.5.**

Let \(f \in B_2(\alpha, \beta), \quad \alpha > 0, \quad -1/2 \leq \beta < 1\). Then the disk \(E\) is mapped onto a domain that contains the disk
\[
|w_0| > \frac{(1-\alpha)(1-\beta) + 2\alpha}{2[(\alpha + 2) + \beta(\alpha - 2)]}.
\]

**Proof.**

From Theorem 3.3.5, it follows that \(f\) is univalent in \(E\). Let \(w_0\) be any complex number such that \(f(z) \neq w_0\) for \(z \in E\). Then
\[
\frac{w_0 f(z)}{w_0 - f(z)} = z + (a_2 + \frac{1}{w_0}) z^2 + ...
\]
is univalent and hence \(a_2 + \frac{1}{w_0} \leq 2\).

Now, using Theorem 2.4.4, we obtain the required result.
2.4.4 A distortion theorem for the class $B_k(I, \beta)$

**Theorem 2.4.6.**

Let $f \in B_k(1, \beta)$. Then, with $|z| = r$, $r_1 = \frac{1 - r}{1 + r}$, we have

\[
\frac{2^{b-1}}{a} \left[ G(a,b,c,-1) - r^2, G(a,b,c,-r_1) \right] \leq |f(z)| \leq \frac{2^{b-1}}{a} \left[ G(a,b,c,-1) - r^{-2}, G(a,b,c,-r_1^{-1}) \right],
\]

where

\[
a = \left( \frac{k}{2} - 1 \right)(1 - \beta) + 1 \quad \text{(2.4.8)}
\]

\[
b = 2\beta \quad \text{(2.4.9)}
\]

\[
c = \left( \frac{k}{2} - 1 \right)(1 - \beta) + 2. \quad \text{(2.4.10)}
\]

**Proof.**

Since $f \in B_k(1, \beta)$, we can write from (2.4.6)

\[
f'(z) = \left( \frac{g(z)}{z} \right)^{1 - \beta}, \quad g \in R_k.
\]

Now, for $g \in R_k$, it is well-known [2] that

\[
\frac{(1 - |z|)^{k-1}}{(1 + |z|)^{k+1}} \leq |g(z)| \leq \frac{(1 + |z|)^{k-1}}{(1 - |z|)^{k+1}}. \quad \text{(2.4.11)}
\]

Therefore, we have

\[
|f'(z)| \geq \frac{(1 - |z|)^{k-1}(1 - \beta)}{(1 + |z|)^{k+1}(1 - \beta)}.
\]

Let $d_r$ denote the radius of the largest schlicht disk centered at the origin contained in the image of $|z| < r$ under $f(z)$. Then there is a point $z_0$, $|z_0| = r$, such that
\[ |f(z_0)| = d_r. \] The ray from 0 to \( f(z_0) \) lies entirely in the image and the inverse image of this ray is a curve \( C \) in \( |z| < r \). Now

\[ d_r = |f(z_0)| = \int_C |f'(z)|dz \]

\[ \geq \int_C \left( \frac{1-|z|}{1+|z|} \right)^{\frac{k}{2}(1-\beta)} dz \]

\[ \geq \int_0^1 \left( \frac{1-s}{1+s} \right)^{\frac{k}{2}(1-\beta)} ds \]

\[ = \int_0^1 \left( \frac{1-s}{1+s} \right)^{\frac{k}{2}(1-\beta)} \frac{ds}{(1+s)^{2(1-\beta)}} \]

\[ \geq \frac{-1}{2} \int_0^1 \left( \frac{1}{1+t} \right)^{\frac{k}{2}(1-\beta)} \left( \frac{2}{1+t} \right)^{2\beta} dt \quad \text{(with } \frac{1-s}{1+s} = t) \]

\[ = (-2)^{2\beta-1} \int_0^{\frac{1-r}{1+r}} \left( \frac{1}{1+t} \right)^{\frac{k}{2}(1-\beta)} (1+t)^{-2\beta} dt \]

\[ + 2^{2\beta-1} \int_0^{\frac{1-r}{1+r}} \left( \frac{k}{2}(1-\beta) \right) (1+t)^{-2\beta} dt \]

\[ = I_1 + I_2. \]

Taking \( \frac{1-r}{1+r} = r_1, \ t = r_1 u \), we have

\[ I_1 = (-2)^{b-1} r_1^a \int_0^1 u^{a-1} (1+r_1 u)^{-b} du \]

\[ = \left( \frac{-2^{b-1}}{a} r_1^a \right) \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} G(a, b, c, -r_1), \]
were \( a, \ b, \ c \) are given by (2.4.8), (2.4.9) and (2.4.10) respectively. We now calculate \( I_2 \).

\[
I_2 = 2^{b-1} \int_0^1 t^{a-1}(1+t)^{-b} dt
\]

\[
= \left( \frac{2^{b-1}}{a} \right) \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} G(a,b,c,-1),
\]

were \( a, \ b, \ c \) are given by (3.3.10), (3.3.11) and (3.3.12) respectively. This gives us the lower bound.

Next we proceed to calculate the upper bound for \( |f(z)| \).

\[
|f'(z)| \leq \frac{(1+|z|)^{\beta-1} \sqrt{z^1}}{(1-|z|)^{\beta-1/2}}.
\]

Therefore

\[
|f(z)| \leq \int_0^1 \frac{(1+s)^{\beta-1/2}}{(1-s)^{\beta-1/2}} ds
\]

\[
\leq (-2)^{2\beta-1} \int_1^{1\;} \xi^{\beta-1/2} (1+\xi)^{-\beta} d\xi
\]

\[
= \frac{2^{b-1}}{a} \left[ G(a,b,c,-1) - r^{-a} G(a,b,c,-r^{-1}) \right],
\]

were \( a, \ b, \ c \) are given by (2.4.8), (2.4.9) and (2.4.10) respectively. This completes the proof.