# Chapter IV

# ON THE LAWS OF FINITE POINTED - GROUPS

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## § 4.1 INTRODUCTION

In this chapter we consider the finite basis problem for the variety generated by a finite pointed group.

A pointed-group may be regarded as a group with an extra nullary operation and so it is an algebra in the sense of universal algebra. Thus our investigation may be viewed in the wider context of the finite basis question for the identities of finite algebras. The main purpose of the developing the theory of pointed-groups is to consider the conjecture of Sheila Oates Macdnonald which states that if $\underline{V}$ is any variety of universal algebras whose congruence lattices are modular then any finite algebra in $\underline{V}$ has a finite basis for its laws. This raises the question: does every finite pointed-group have a finite basis for its laws? and is the analogous statement to the well-known Theorem of Oates and Powell which states that every finite group has a finite basis for its laws (see [13]).

Now if we can find a finite pointed-group which does not have finite basis for its laws, then the conjecture of Sheila Oates Macdnald and thus the generalization of Oates and Powell Theorem to pointed-groups would be false, because as we shall see below, that the congruence lattice of any pointed-group is modular.

In order to establish this latter fact we shall need some elementary properties of universal algebra. These are well-known results but are described here for the sake of completeness.

Let S be a set and let n be a positive integer. We write $S^n$ to denote the Cartesian product SxSx ... xS of n-copies of S.

**DEFINITION**: For a given set S and a natural number n, any function from $S^n$ to S is called an *n-array operation* on S.

A *0-ary operation* or (nullary operation) on S is called an element of S.

If $\omega$ is an n-ary operation on S for some $n \geq 0$, then $\omega$ is called an operation on S.

**DEFINITION**: A set S together with a set $\Omega$ of operations on S is called *universal algebra*.

**EXAMPLES**: i. A group is a universal algebra with *one 2-ary or binary* operation i.e. $(g,h) \mapsto gh$, *one 1-ary or uninary* operation i.e. $g \mapsto g^{-1}$ and one *0-ary or nullary* operation i.e., the identity element.

ii. A pointed-group is a group with an extra nullary operation.

Suppose S is a universal algebra with $\Omega$ as its set of operations. A subset T of S is called a sub algebra of S, if T is closed under all elements of $\Omega$ i.e., if T contains every nullary operation of $\Omega$ and for every n-ary-operation $\omega$ of $\Omega$ with $n \geq 1$ we have

$$(T^n)\ \omega \subseteq T$$

Clearly every sub algebra of a universal algebra S is itself a universal algebra. Also, S x S can be given structure of a universal algebra by using the elements of $\Omega$ to define operations on S x S componentwise. In particular if $\omega$ is a nullary operation on S, then $(\omega , \omega)$ is the corresponding nullary operation on S x S.

**DEFINITION**: An equivalence relation on a universal algebra S which is also a sub algebra of S x S is called a *congruence* on S.

In particular, a congruence on a group G is an equivalence relation on G which is a sub group of G x G.

Suppose q is a congruence on a group G. We write $x_q$ where $x \in G$, to denote the q-class of x i.e., the sub set of G defined by

$$x_q = \{ y \in G \mid (x,y) \in q\}$$

Since, q is reflexive we have $x \in x_q$. In fact $(x,y) \in q$ if and only if $x_q = y_q$.

**4.1.1** __LEMMA__: For a congruence q on a group G, let $1_q$ be the equivalence class containing 1 i.e. $1_q = \{x \in G \mid (1, x) \in q\}$. Then $1_q$ is a normal sub group of G.

**Proof:** First we shall prove that $1_q$ is a sub group of G. $1_q$ is non-empty because $1 \in 1_q$. Let $t_1$ , $t_2 \in 1_q$. then by definition $(1 , t_1) \in q$ and $(1 , t_2) \in q$. But q is a sub group of G x G. Therefore $(1 , t_1t_2) \in q$. Thus $t_1t_2 \in 1_q$. Again let $t \in 1_q$, then $(1 , t) \in q$. Since q is a sub group of G x G, so $(1 , t)^{-1} \in q$ i.e., $(1 , t^{-1}) \in q$. Therefore $t^{-1} \in 1_q$. Hence $1_q$ is a sub group of G as desired.

Next we shall show that $1_q$ , is a normal sub group of G. Let $t \in 1_q$ and let g be any element of G. Now to prove that $1_q$ is a normal sub group of G, it is enough to show that $g^{-1}tg \in 1_q$. To do this it is enough to show that $(1 , g^{-1}tg) \in q$. But this holds, because q is a sub group of GxG and $(1 , g^{-1}tg) = (g^{-1} , g^{-1})(1 , t) (g , g)$ belongs to q. Thus $1_q$ is a normal sub group of G as required.

**4.1.2** **LEMMA:** Let q be a congruence on a group G and let $g_1$ and $g_2$ be a pair of elements of G. Then $(g_1, g_2) \in q$ if and only if $g_1$ and $g_2$ lie in the same coset of $1_q$ in G i.e. $g_1^{-1}g_2 \in 1_q$.

**Proof:** $(g_1, g_2) \in q$ if and only if $(g_1^{-1}, g_1^{-1})(g_1, g_2) \in q$ i.e. if and only if $(1, g_1^{-1}g_2) \in q$ i.e. if and only if $g_1^{-1}g_2 \in 1_q$ i.e. if and only if $g_1$ and $g_2$ lie in the same coset of $1_q$ in G.

**4.1.3** **LEMMA:** Suppose N is a normal sub group of G. Let q be the sub set of G x G consisting of all elements $(g_1, g_2)$ such that $g_1^{-1}g_2 \in N$. Then q is a congruence on G.

**Proof:** To prove that q is a congruence on G, it is enough to show that q is an equivalence relation on G and q is a sub group of G x G.

But this is clear for $g_1, g_2 \in G$, the relation $g_1 \sim g_2$ if and only if $g_1^{-1}g_2 \in N$ i.e., if and only if $g_1$ and $g_2$ lie in the same coset of N is an equivalence relation on G. Next we shall show that q is a sub group of G x G.Now q is non empty, because $(1, 1) \in q$. Let $(g_1, g_2) \in q$ and $(h_1,h_2) \in q$. To show that q is a sub group of G x G it is enough to show that $(g_1, g_2)^{-1}(h_1,h_2)=(g_1^{-1}, g_2^{-1})(h_1,h_2)=(g_1^{-1}h_1^{-1}, g_2 h_2) \in q$. To do this, it is enough to show that $(g_1^{-1}h_1)^{-1}(g_2^{-1}h_2) \in N$.

Now $g_1^{-1}g_2 \in N$ and $h_1^{-1}h_2 \in N$. Thus $(g_1^{-1}g_2)^{-1} = g_2^{-1}g_1 \in N$. Since N is a normal sub group of G, we have

$$(g_2^{-1}h_2)^{-1}(g_2^{-1}g_1)(g_2^{-1}h_2) = (h_2^{-1}g_1g_2^{-1}h_2) \in N.$$

Therefore, $(h_1^{-1}h_2)(h_2^{-1}g_1g_2^{-1}h_2) = h_1^{-1}g_1g_2^{-1}h_2$

$$= ((g_1^{-1}h_1)^{-1})(g_2^{-1}h_2) \in N \text{ as}$$

required. Therefore, q is a congruence on G.

It is now immediate from Lemmas 4.1.2 and 4.1.3 that there is a one-to-one correspondence between the set of congruences of a group G and the set of normal sub groups of G.

Now suppose that q is a congruence on a group G. Let $c \in G$. Then $(c,c) \in q$, because q is reflexive. Therefore, (q, (c,c)) is a sub pointed-group of (G,c)x(G,c). Thus q is congruence on (G,c) also.

Conversely, any congruence on a pointed-group (G,c) is a congruence on its carrier G. Thus it follows that the congruences of a pointed-group (G,c) are the same as those of the group G. Therefore, there is a one-to-one correspondence between the set of congruences of a pointed-group and the set of its normal sub groups.

**DEFINITION:** A set S with a binary relation $\leq$ is called a *partially ordered set* if it satisfies the following conditions:

i. $s \leq s$ for all $s \in S$.

ii. If $s_1 \leq s_2$ and $s_2 \leq s_1$ then $s_1 = s_2$ for all $s_1, s_2 \in S$.

iii. If $s_1 \leq s_2$ and $s_2 \leq s_3$ then $s_1 \leq s_3$ for all $s_1, s_2, s_3 \in S$.

Suppose S is a partially ordered set and T is a sub set of S. Suppose that $s \in S$ satisfies $s \geq t$ for all $t \in T$. Then s is called an upper bound of T. Similarly we define a lower bound of T as an element $s \in S$ such that $t \geq s$ for all $t \in T$.

An upper bound s of T is called a *least upper bound* if when ever s′ is an upper bound of T then $s' \geq s$. Similarly a lower bound s of T is called a *greatest lower bound* if when ever s′ is a lower bound of T then $s \geq s'$.

Thus it follows from these definitions that if T has a least upper bound then it is unique and similarly if T has a greatest lower bound then it is unique.

**DEFINITION:** A partially ordered set S is called a *lattice* if every pair $s_1, s_2$ of elements of S has a greatest lower bound and a least upper bound. The greatest lower bound and the least upper bound of $\{s_1, s_2\}$ in S are denoted by $s_1 \wedge s_2$ and $s_1 \vee s_2$ respectively. $s_1 \wedge s_2$ sometimes called the *meet* of $s_1$ and $s_2$ and $s_1 \vee s_2$ is sometimes called the *join* of $s_1$ and $s_2$.

**DEFINITION:** A lattice S which satisfies:

i. $a \geq c \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee c$ for all $a,b,c \in S$, is called a *modular lattice*.

The important example of a modular lattice in elementary group theory is that the normal sub groups of a group form a modular lattice under the inclusion relation. we omit the (easy) proof. it is sufficient to say that the meet of normal sub groups H and K is $H \cap K$ and the join of H and K is HK.

Let $\mathcal{C}$ be the set of all congruences of a universal algebra S with a set of operations $\Omega$. Then each element of $\mathcal{C}$ is a sub set of $S \times S$. Therefore, $\mathcal{C}$ is a partially ordered set under the rlation $\leq$ of inclusion.

Let $p, q \in \mathcal{C}$. We define relations $p \wedge q$ and $p \vee q$ on S as follows:

$p \wedge q$ is defined as the interesction $p \cap q$ of p and q, while $p \vee q$ consists of all elements (s,t) of $S \times S$ such that there exists a finite sequence of elements $s_o, s_1, s_2, \dots s_n$ of S for which $s = s_o$, $t = s_n$ and $(s_i, s_{i+1}) \in p$ or $(s_i, s_{i+1}) \in q$ for $i = 0, \dots, n-1$.

A proof of the following result may be found in [6].

**4.1.4** **THEOREM**: The set $\mathcal{C}$ of all congruences of universal algebra S is a lattice under the relation of inclusion. (This lattice is called the congruence lattice of S). If p, q belong to $\mathcal{C}$ then the meet $p \wedge q$ and join $p \vee q$ of $p,q$ in $\mathcal{C}$ are relations.

Thus in particular, the set of all congruences of a pointed-group $(G,c)$ forms a lattice under the partially ordered relation $\leq$ of inclusion.

**DEFINITION:** A one-to-one mapping $\alpha$ from a lattice $L_1$ on to a Lattice $L_2$ is called a lattice isomorphism, if for all $a, b \in L_1$ we have $a \leq b$ if and only if $a\alpha \leq b\alpha$.

Suppose $\alpha$ is a lattice isomorphism from $L_1$ to $L_2$. Then the inverse function $\alpha^{-1} : L_2 \to L_1$ is also an isomorphism. If there is a lattice isomorphism between two lattices $L_1$ and $L_2$, then we say that $L_1$ and $L_2$ are isomorphic and write $L_1 \cong L_2$.

**4.1.5 THEOREM:** The lattice of congruences of a pointed-group $(G,c)$ is isomorphic to the lattice of its normal sub groups.

**Proof:** Let $\mathcal{M}$ denote the lattice of normal sub groups of G and let $\mathcal{C}$ denote the lattice of congruences of $(G,c)$. Let $\alpha : \mathcal{M} \to \mathcal{C}$ be the mapping defined by

$$N\alpha = q_N \quad \text{for all } N \in \mathcal{M}$$

where

$$q_N = \{(x,y) \quad x^{-1}y \in N\}.$$

Clearly Lemmas 4.1.1, 4.1.2, and 4.1.3 essentially prove that $\alpha$ is a one-to-one mapping from $\mathcal{M}$ to $\mathcal{C}$. Now to prove that $\alpha$ is an isomorphism, it only remains to prove that for any $N_1, N_2 \in \mathcal{M}$ we have $N_1 \leq N_2$ if and only if $N_1\alpha \leq N_2\alpha$.

So let $N_1 \leq N_2$. Then to prove that $N_1\alpha \leq N_2\alpha$, we need to show that $q_{N_1} \leq q_{N_2}$. So let $(x,y) \in q_{N_1}$. Then $x^{-1}y \in N_1$. But $N_1 \leq N_2$. Thus $x^{-1} y \in N_2$. Therefore, $(x,y) \in q_{N_2}$. Thus $q_{N_1} \leq q_{N_2}$.

Conversely, let $N_1\alpha \leq N_2\alpha$ i.e., $q_{N_1} \leq q_{N_2}$. To prove that $N_1 \leq N_2$, it is enough to show that every element of $N_1$ belongs to $N_2$.

So, let $x \in N_1$, then $(1,x) \in q_{N_1}$. But $q_{N_1} \leq q_{N_2}$. Thus $(1,x) \in q_{N_2}$. Therefore $x \in N_2$ as desired. Thus $\alpha$ is an isomorphism. Hence $\mathcal{M} \cong \mathcal{C}$.

Further, since we have already noted that the lattice $\mathcal{M}$ of normal sub group of G is modular, it follows, from Theorem 4.1.5, that the lattice $\mathcal{C}$ of congruences of a pointed-group $(G,c)$ is modular.

## § 4.2 SOME FINITELY BASED VARIETIES OF GROUP

In this and in the next sections we shall consider the question of whether the laws of every finite pointed-group are finitely based. A negative answer would provide a counter example to the conjucture of Sheila Oates Macdonald mentioned earlier in the previous section. On the other hand, a positive answer would give a generalization of the theorem in [13] of Sheila Oates and M.B. Powell which says that the laws of every finite group are finitely based. Fortunately we have been able to resolve the question with negative answer as we provide a counter example in chapter 5.

In the next section we shall consider the pointed-groups (G,c) where G is a dihedral group of order 6 and c ranges over the elements of G. Since G is metabelian, the laws of these pointed-groups are finitely based by Theorem 3.3.1. Nevertheless, Theorem 3.3.1 does not give any idea of how to find finite basis. In order to give some insight into the laws of finite pointed-groups we shall find finite basis for these pointed-groups.

In this section we lay foundations for this by finding a finite basis for the laws of the group G. All that this section contains is well known. For example considerably more general results are to be found in John Cossey's Ph.D thesis. But, I have not had access to this and have had to establish the results myself.

For each positive integer n we write $\underline{\underline{A}}_n$ to denote the variety of all abelian groups of exponent dividing n. Also, $\underline{\underline{A}}_m \underline{\underline{A}}_n$ denotes the product variety consisting of all groups H which have a normal sub group N such that $N \in \underline{\underline{A}}_m$ and $H/N \in \underline{\underline{A}}_n$. Notice that these are varieties of groups. Laws of groups will always be taken as words $w(x_1, x_2, \ldots, x_n)$ in the variables $x_1, x_2, \ldots, x_n$. The first result is:

**4.2.1** **<u>THEOREM</u>:** The variety $\underline{\underline{A}}_3 \underline{\underline{A}}_2$ is finitely based and has a finite basis $\{x_1^6, [x_1^2, x_2^2]\}$.

**Proof:** To prove the theorem, it is enough to show that $\underline{\underline{A}}_3 \underline{\underline{A}}_2 = \mathrm{var}\{x_1^6, [x_1^2, x_2^2]\}$. To do this it is enough to show that:

i. $\underline{\underline{A}}_3 \underline{\underline{A}}_2 \subseteq \mathrm{var}\{x_1^6, [x_1^2, x_2^2]\}$.

ii. $\mathrm{var}\{x_1^6, [x_1^2, x_2^2]\} \subseteq \underline{\underline{A}}_3 \underline{\underline{A}}_2$.

Now suppose that $A \in \underline{\underline{A}}_3 \underline{\underline{A}}_2$. Then by the definition of a product variety there exists a normal sub group N of A such that $N \in \underline{\underline{A}}_3$ and $A/N \in \underline{\underline{A}}_2$. Our aim is to show that A belongs to the var $\{x_1^6, [x_1^2, x_2^2]\}$. Now since $A/N \in \underline{\underline{A}}_2$, so $x_1^2$ and $[x_1, x_2]$ are laws of A/N. Let $a_1, a_2 \in A$. Then $[a_1, a_2] \in N$ and $a_1^2 \in N$, because $x_1^2$ and $[x_1, x_2]$ are laws of A/N. Since $N \in \underline{\underline{A}}_3$, so we have $(a_1^2)^3 = 1$ and $[a_1^2, a_2^2] = 1$. Therefore, A belongs to the var $\{x_1^6, [x_1^2, x_2^2]\}$. Thus $\underline{\underline{A}}_3 \underline{\underline{A}}_2 \subseteq \mathrm{var}\{x_1^6, [x_1^2, x_2^2]\}$.

To complete the proof it only remains to prove that $\mathrm{var}\{x_1^6, [x_1^2, x_2^2]\} \subseteq \underline{\underline{A}}_3 \underline{\underline{A}}_2$. To do this it is enough to show that every group in the variety generated by $x_1^6$ and $[x_1^2, x_2^2]$

belongs to $\underline{\underline{A}}_3\ \underline{\underline{A}}_2$.

Now suppose that $A \in \text{var}\ \{x_1^6, [x_1^2, x_2^2]\}$. Then $x_1^6$ and $[x_1^2, x_2^2]$ are laws of A. Now to prove that $A \in \underline{\underline{A}}_3\ \underline{\underline{A}}_2$, it is enough to show that A has a normal sub group N such that $N \in \underline{\underline{A}}_3$ and $A/N \in \underline{\underline{A}}_2$.

Let $N = \{x_1^2\}(A)$ be the word sub group of A corresponding to the word $x_1^2$. Then $A/N \in \text{var}\ \{x_1^2\}$. Therefore A/N is an abelian group of exponent dividing 2. Therefore $A/N \in \underline{\underline{A}}_2$ as required. Hence, to finish the proof, it only remains to prove that $N \in \underline{\underline{A}}_3$. To do this it is enough to show that N is an abelian group of exponent 1 or 3. Now N is the sub group of A generated by the squares of the elements of A i.e. $N = < a^2 \mid a \in A >$. To show that N is abelian, it is enough to show that the generators $a^2$ of N commute. But this follows because A has the law $[x_1^2, x_2^2]$. Also, the law $x_1^6$ of A shows that the generators of N are of order dividing 3. Now since N is abelian, it follows that N has exponent dividing 3. Therefore $N \in \underline{\underline{A}}_3$. Thus $A \in \underline{\underline{A}}_3\ \underline{\underline{A}}_2$, as required.

Therefore, $\text{var}\ \{x_1^6, [x_1^2, x_2^2]\} \subseteq \underline{\underline{A}}_3\ \underline{\underline{A}}_2$. Hence $\underline{\underline{A}}_3\ \underline{\underline{A}}_2 = \text{var}\ \{x_1^6, [x_1^2, x_2^2]\}$, as desired. Thus $\underline{\underline{A}}_3\ \underline{\underline{A}}_2$ is finitely based variety and has finite basis, $\{x_1^6, [x_1^2, x_2^2]\}$.

**4.2.2** <u>**THEOREM**</u>: Let G be a dihedral group of order 6. Then var $G = \underline{\underline{A}}_3\ \underline{\underline{A}}_2$.

**Proof:** Let $G = \langle a, b \mid a^3 = b^2 = 1, b^{-1} ab = a^{-1} \rangle$. Then G is a non abelian group with the property that every proper sub group and every proper factor group of G is abelian. Therefore G, is a critical group. (A group is called *critical* if it is finite and does not belong to the variety generated by its proper sub groups and proper factor groups). Hence G is minolithic. (A group with a unique non trivial minimal normal sub group is called *monolithic*).

Clearly the unique minimal normal sub group of G is the sub group $\langle a \rangle$ of order 3. We want to prove that var-$G = \underline{\underline{A}}_3 \underline{\underline{A}}_2$. To do this, we will have to show that:

i. var $G \subseteq \underline{\underline{A}}_3 \underline{\underline{A}}_2$

ii. $\underline{\underline{A}}_3 \underline{\underline{A}}_2 \subseteq$ var $G$

Now $\langle a \rangle \in \underline{\underline{A}}_3$ and $G/\langle a \rangle \in \underline{\underline{A}}_2$. Thus $G \in \underline{\underline{A}}_3 \underline{\underline{A}}_2$. It follows that var $G \subseteq \underline{\underline{A}}_3 \underline{\underline{A}}_2$. To complete the proof, it only remains to show that var G contains $\underline{\underline{A}}_3 \underline{\underline{A}}_2$ i.e. var $G \supseteq A_3 A_2$. To do this, we shall need to prove some subsidiary results.

**4.2.3** **<u>LEMMA</u>:** Let M be a monolithic group in $\underline{\underline{A}}_3 \underline{\underline{A}}_2$. Then either M is cyclic of order 2 or 3 or M is a dihedral group of order 6.

**Proof:** Let $M \in \underline{\underline{A}}_3 \underline{\underline{A}}_2$. Then there exists a normal sub group N of M such that $N \in \underline{\underline{A}}_3$ and $M/N \in \underline{\underline{A}}_2$.

Now to prove the Lemma we have to consider the following three cases:

<u>Case (i)</u> Suppse $N = \{1\}$. Then $M \in \underline{\underline{A}}_2$. Therefore, M is the direct product of a finite number of cyclic groups of order 2. But M is monolithic. Thus M is cyclic of order 2.

<u>Case (ii)</u> Suppose $N = M$. then $M \in \underline{\underline{A}}_3$. Therefore, by the same argument as in case (i), M is cyclic of order 3.

<u>Case(iii)</u> Suppose $\{1\} < N < M$. Now since $N \in \underline{\underline{A}}_3$ and $M/N \in \underline{\underline{A}}_2$, therefore, $N \cong C_3 x \ldots x C_3$ and $M/N \cong C_2 x \ldots x C_2$ (where $C_n$ denotes a cyclic group of order n).

Now first we want to show that N has a complement in M i.e. there is a sub group H of M such that $M = NH$ and $N \cap H = \{1\}$.

Let $|M| = 3^\alpha 2^\beta$. Then $|M| = 3^\alpha$ and $|M/N| = 2^\beta$. Let H be a sylow sub group of M corresponding to a prime 2. Then H has order $2^\beta$. Thus $N \cap H = \{1\}$. Now to show that H is a complement of N in M, it only remains to show that $M = NH$. Since $N \trianglelefteq M$ and $H \leq M$, so NH is a sub group of M. But since N and H are sub groups of NH, so NH has order $3^\alpha 2^\beta = |M|$. Thus $NH = M$ as desired.

Next we want to show that $\mathcal{C}_M(N) = N$ where $\mathcal{C}_M(N)$ denotes the centralizer of N in M. Since N is abelian, so $N \leq \mathcal{C}_M(N)$. Thus, since $M=NH$, we can write:

$$\mathcal{C}_M(N) = N\ (H \cap \mathcal{C}_M\ (N))$$

Thus our aim is to show that $H \cap \mathcal{C}_M\ (N) = \{1\}$. Now

$H \cap \mathcal{C}_M(N)$ is centralized by $N$ and normal in $H$. Hence, $H \cap \mathcal{C}_M(N)$ is normal in $M=NH$. Also, $H \cap \mathcal{C}_M(N) \cap N \leq H \cap N$. But $H \cap N = \{1\}$. Thus, as $M$ is monolithic, we have $H \cap \mathcal{C}_M(N) = \{1\}$ and so $\mathcal{C}_M(N) = N$ as desired.

Further we want to show that $N$ is a minimal normal sub group of $M$. To do this, we see that for each $h \in H$ the mapping $n \mapsto n^h$ where $n \in N$, is an automorphism of $N$, because $N$ is a normal sub group of $M$. Therefore, each element of $H$ acts by conjugation as an automorphism of $N$. Now since $N$ is an elementary abelian 3-group. Therefore, if we write the group operation in $N$ additively, then $N$ can be regarded as a vector-space over the field of 3 elements. In this terminology the elements of $h$ act as linear transformations of $N$. Hence $N$ can be regarded as a module for $H$ over the field of 3 elements.

As mentioned above, our aim is to show that $N$ is a minimal normal sub group of $M$. suppose otherwise that there is a normal sub group $N_1$ of $M$ such that $\{1\} < N_1 < N$. Now since $N_1$ is normal in $M$, so $N_1$ is $H$-invariant i.e., $n_1{}^h \in N_1$ for all $h \in H$, $n_1 \in N_1$. Therefore $N_1$ is an $H$-sub module of $N$. Now by Maschke's Theorem [Theorem 15.6 of (4)], there is an $H$-sub module $N_2$ of $N$ such that $N=N_1 \oplus N_2$ [or, in the multiplication notation, $N_2$ is an $H$-invariant sub group of $N$ such that $N = (N_1 \times N_2)$].

Now since $N$ is abelian, $N_2$ is normal in $N$. Also, $N_2$ is $H$-invariant. Therefore, $N_2$ is normal in $M = NH$. Therefore,

$N_1$ and $N_2$ are two non-trivial normal sub groups of M such that $N_1 \cap N_2 = \{1\}$. This contradicts the fact that M is monolithic.

The argument used above shows that N is an irreducible module for H. Thus N is a minimal normal sub group of M. In fact N is the unique minimal normal sub group of M because M is monolithic. Moreover N is a faithful H-module. In other words no non trivial element of h acts as an identity transformation on N. This is because $\mathcal{C}_M(N) = N$ i.e. no non trivial element of H centralizes N. Thus N is a faithful irreducible module for H. Therefore, by Theorem 3.2.2 of [5], Z(H) is cyclic. But H is abelian. Therefore, H is cyclic. Since $H \in \underline{\underline{A}}_2$, so H is cyclic of order 2.

We shall now show that N is cyclic of order 3. We have shown that H is cyclic of order 2. So let $H = \langle h \rangle$. We want to show that N is cyclic. To do this first we show that the identity element of N is the only element of N which is centralized by H. In order to get a contradiction we assume otherwise that $a \in N \setminus \{1\}$ and that a is centralized by H. Therefore $\langle a \rangle$ is a normal sub group of M. Now, since N is the monolith of M, (in a monolithic group, the unique non-trivial minimal normal sub group is called the monolith), we have $\langle a \rangle = N$. Thus $\mathcal{C}_M(\langle a \rangle) = M$. But this contradicts the fact that $\mathcal{C}_M(N) = N$.

Further we show that if a is any element of N, then $aa^h$ is centralized by h. This follows because

$$\begin{aligned} h^{-1}(aa^h)h &= h^{-1}(ah^{-1}ah)h \\ &= (h^{-1}ah)(h^{-1}h^{-1}ah)h \\ &= (h^{-1}ah)(h^{-2}ah^2) \\ &= a^h a^{h^2} \\ &= a^h a \quad \text{(as H is cyclic of order 2)} \\ &= aa^h \quad \text{(as N is abelian)} \end{aligned}$$

Therefore $aa^h$ is centralized by h. But we have shown above that only the identity element of N is centralized by H. Thus $aa^h = 1$ i.e. $a^h = a^{-1}$ for all $a \in N$.

Now take $a \in N\backslash\{1\}$. Then since $a^h = a^{-1}$, so $<a>$ is a normal sub group of M. Since N is the monolith of M, it follows that $N = <a>$. Thus N is cyclic of order 3. Now we have $M = NH$ where N is cyclic of order 3, H is cyclic of order 2, $h^{-1}ah = a^{-1}$, $N = <a>$ and $H = <h>$. Therefore, M is dihedral group of order 6. This completes the proof of the Lemma 4.2.3.

From the above Lemma, it follows that if M is a monolithic group in $\underline{\underline{A}}_3\ \underline{\underline{A}}_2$, then either M is cyclic of order 2 or 3 or M is a dihedral group of order 6. In every case $M \in$ var G.

**4.2.4** **<u>COROLLARY</u>**: Let A be a finite group in $\underline{\underline{A}}_3\ \underline{\underline{A}}_2$. Then $A \in$ var G, where G is a dihedral group of order 6.

**Proof:** Since A is a finite group, so by (51,43) of [12], A belongs to the variety generated by its critical factors. But the critical factors of A belong to $\underline{\underline{A}}_3\ \underline{\underline{A}}_2$ because A $\in \underline{\underline{A}}_3\ \underline{\underline{A}}_2$. Therefore, by lemma 4.2.3, the critical factors of A belong to var G. Thus A $\in$ var G as required.

The following result will complete the proof of the Theorem 4.2.2.

**4.2.5** <u>COROLLARY</u>: Let A be any group in $\underline{\underline{A}}_3\ \underline{\underline{A}}_2$. Then A $\in$ var G where G is a dihedral group of order 6.

**Proof**: To prove that A $\in$ var G it is enough to show that every finitely generated sub group of A belongs to var G. To do this it is enough to show that every finitely generated group in $\underline{\underline{A}}_3\ \underline{\underline{A}}_2$ is finite, because the result will then follow from Corollary 4.2.4.

Thus we need to prove that $\underline{\underline{A}}_3\ \underline{\underline{A}}_2$ is a locally finite variety. (A variety is *locally finite* if all of its finitely generated groups are finite).

Now let B be a finitely generated group in $\underline{\underline{A}}_3\ \underline{\underline{A}}_2$. Then there exists a normal sub group N of B such that N $\in \underline{\underline{A}}_3$ and B/N $\in \underline{\underline{A}}_2$. Now to prove that B is finite it is enough to show that N and B/N are finite. Since B is a finitely generated group, B/N is finitely generated. But B/N $\in \underline{\underline{A}}_2$. Therefore, B/N is finite. Now since B/N is finite, N has a finite index in B. But, B is finitely generated group. Therefore, by Corollary 7.2.1 of [7], N is finitely generated.

Therefore N is finite. Thus B is finite as required.

We have now completed the proof of Theorem 4.2.2. An immediate consequence of the Theorems 4.2.1 and 4.2.2 is the following:

**4.2.6** **COROLLARY**: The variety gnerated by a dihedral group of order 6 has a finite basis $\{x_1^6, [x_1^2, x_2^2]\}$ for its laws.

## § 4.3 THE LAWS OF SOME FINITE POINTED-GROUPS:

**DEFINITION**: Let G be a group. Let c,d be elements of G. The elements c and d of G are said to be *similar* (written as $c \sim d$) if there is an automorphism $\alpha$ of G such that $c\alpha = d$.

Obviously, similar elements of a group G have the same order. Also clearly similarity is an equivalence relation on G. Therefore, the whole group G is partitioned with respect to the automorphisms of G into disjoint similarity classes.

**4.3.1** **THEOREM**: Suppose G is a group. Let c and d be similar elements of G. Then var(G,c) = var(G,d) i.e. (G,c) and (G,d) have the same laws.

**Proof**: Let $\alpha$ be an automorphism of G such that $c\alpha = d$. Then $\alpha$ is a pointed-group isomorphism from (G,c) to (G,d). Thus

$(G,c) \cong (G,d)$ and certainly $\text{var}(G,c) = \text{var}(G,d)$ i.e. $(G,c)$ and $(G,d)$ have the same laws.

By the above theorem 4.3.1 in studying the laws of pointed-groups with carrier G it is enough to consider the pointed-groups $(G,c)$ where c ranges over representatives of the similarity classes in G.

**4.3.2** **LEMMA**: Let $G= \langle a,b \mid a^3 = b^2 = 1,\ b^{-1}ab = a^{-1}\rangle$ be a dihedral group of order 6. Then the similarity classes in G are $\{1\}$, $\{a,a^2\}$ and $\{b,ab,a^2b\}$.

**Proof:** The given group G consists of six elements namely $\{1,a,a^2, b,ab,ab^2\}$ with two elements nalely $a,a^2$ of order 3 and with 3 elements namely $b,ab,a^2b$ of order 2. Clearly any two elements of G which come from different subsets in the collection $\{1\}$, $\{a,a^2\}$ and $\{b,ab,a^2b\}$ cannot be similar because they have different orders. Thus to prove the lemma it is sufficient to prove that:

i. $a \sim a^2$

ii. $b \sim ab$

and iii. $b \sim a^2b$

Now $I_b$, $I_a$, $I_{a^2}$ are the inner automorphisms such that $(a)I_b = a^2$, $(b)I_{a^2} = ab$ and $(b)I_a = a^2b$. Thus the similarity classes in G are $\{1\}$, $\{a, a^2\}$, $\{b, ab, a^2b\}$ as desired.

**REMARKS**: If (G,c) is a pointed-group, then the laws of G are elements of the free group F on $x_1, x_2, \ldots,$ while the laws of (G,c) are elements of the free group X on $y, x_1, x_2, \ldots$. Clearly we may regard F as a sub group of X. Hence the laws of G are elements of (X,y).

**4.3.3** **THEOREM**: Let G be a group. Then the laws of (G,1) have a basis consisting of the laws of G together with the word y.

**Proof:** To prove the theorem, it is enough to show that the set of laws of (G,1) is the closure of the set of words consisting of the laws of G together with the word y.

Let V be the set of laws of G and let W be the set of laws of (G,1). Clearly $V \cup \{y\} \subseteq W$. Now let $w(y,x_1,x_2,\ldots,x_n)$ be any law of (G,1). To complete the proof, it only remains to show that $w(y,x_1,x_2,\ldots,x_n)$ is a consequence of $V \cup \{y\}$. To do this, we may write.

$$w(y,x_1,x_2,\ldots, x_n) = u.v(x_1,\ldots, x_n)$$

where u is a product of conjugates of $y^{\pm 1}$ and where $v(x_1,x_2,\ldots,x_n)$ is an element of F. Now clearly u is a consequence of y. The proof will be completed if we can show that $v(x_1,x_2,\ldots,x_n)$ is a law of G i.e. $v \in V$.

To do this let $\alpha$ be any homomorphism from F to G. Then $\alpha$ may be extended to a homorphism $\bar{\alpha} : (X,y) \to (G,1)$ by defining $y\bar{\alpha} = 1$.

Now since $w(y,x_1,\ldots, x_n)$ is a law of $(G,1)$ so we have,

$$w(y,x_1,\ldots,x_n)\alpha = 1$$

i.e. $u\alpha.v(x_1,\ldots,x_n)\alpha = 1$. Clearly $u\alpha = 1$. Thus $v(x_1,x_2,\ldots,x_n)\alpha = 1$. But $v(x_1,\ldots,x_n)\alpha = v(x_1,\ldots,x_n)\alpha$. Hence $v(x_1,\ldots,x_n)\alpha = 1$ and therefore, $v(x_1,x_2,\ldots,x_n)$ is a law of G as desired. Thus the theorem follows.

The following theorem is a generalization of the Theorem 4.3.3.

**4.3.4** <u>THEOREM</u>: Let G be a group. Let c be an element of the centre of G. Then the laws of $(G,c)$ have a basis consisting of the laws of G together with the laws $[y , x_1]$ and $y^n$ where n is the order of c taking $n=0$ if c has infinite order.

**Proof:** To prove the theorem, it is enough to show that the set of laws of $(G,c)$ is the closure of the set of words consisting of the laws of G together with the laws $y^n$ and $[y , x_1]$.

Let V be the set of laws of G and let W be the set of all laws of $(G,c)$. Then clearly $V \subseteq W$. Also, for any homomorphism $\alpha: (X,y) \rightarrow (G,c)$ we have $y\alpha=c$. Thus $y^n$ is a law of $(G,c)$ i.e. $y^n \in W$. Again, since c belongs to the centre of G, so c commutes with every element of G i.e. $[c , g_1] =1$ for all $g_1 \in G$. Thus $[y , x_1]$ is a law of $(G,c)$ i.e. $[y , x_1] \in W$.

Now let $w = w(y,x_1,\ldots,x_n)$ be any law of $(G,c)$. To complete the proof, it only remains to show that $w$ is a consequence of $\{y^n, [y , x_1]\} \cup V$. To do this, we shall first show that $w$ can be written in the form

$$w = w_1 w_2 w_3 \qquad \text{(i)}$$

where $w_1 = y^m$ for some integer $m$, $w_2$ is a product of elements of the form $[v, y^{\pm 1}]^u$ $(u, v \in X)$ and $w_3$ is an element of $F$.

Now using the identities

$$vy = yv[v , y]$$

and

$$vy^{-1} = y^{-1}v[v , y^{-1}]$$

we can write $w$ in the form

$$w = y^m w'$$

where $w'$ is a product of elements of the form $[v, y^{\pm 1}]$ $(v \in X)$ and the elements of $F$.

Again using the identity

$$u[v , y^{\pm 1}] = [v , y^{\pm}]^{u-1} u ,$$

we can write $w'$ in the form

$$w' = w_2 w_3$$

where $w_2$ is a product of elements of the form $[v , y^{\pm 1}]^u$ $(u , v \in X)$ and $w_3$ is an element of $F$.

Thus we have shown that $w$ can be written in the required form (i). Now we want to show that $n$ divides $m$ because it will then follow that $w_1$ is a consequence of $y^n$. To do this, substituting $c$ for $y$ and $1$ for all $x_1, x_2, \ldots, x_n$ in (i), $w_1$ takes the value $c^m$, $w_2$ takes the value $1$, because $c$ belongs to the centre of $G$, and $w_3$ takes the value $1$ because $w_3 \in F$. Thus the word $w$ takes the value $c^m$. But $w$ is a law of

$(G,c)$. Hence $c^m=1$. It follows that $n$ divides $m$, because $c$ has order $n$. Thus $w_1$ is a consequence of $y^n$. Also, clearly $w_2$ is a consequence of $[y\ ,\ x_1]$. Since $w$ is a law of $(G,c)$ and $w_1,w_2$ are laws of $(G,c)$ it follows that $w_3$ is a law of $(G,c)$. But $w_3 \in F$. Hence $w_3 \in V$. Thus $w$ is a consequence of $y^n$, $[y,x_1]$ and $V$ as desired. This proves the theorem.

By using theorem 4.3.4 we shall now show by an example that the converse of Theorem 4.3.1 is false.

**EXAMPLE**: Let $G = \langle a,b \mid a^4 = b^2 = 1,\ bab^{-1} = a\rangle$. Thus $G$ is a non-cyclic abelian group of order 8. Take the elements $a^2$ and $b$ of $G$. Now $a^2$ and $b$ have the same order 2. Also $a^2$ and $b$ belong to the centre of $G$. Hence, by Theorem 4.3.4, the laws of $(G,a^2)$ and $(G,b)$ have a basis consisting of the laws of $G$ together with the laws $y^2$ and $[y,x_1]$. Thus we have

$$\mathrm{var}(G,a^2) = \mathrm{var}(G,b)$$

But $a^2$ is not similar to $b$, because we shall now show that there is no automorphism $\alpha$ of $G$ such that $a^2\alpha = b$. To get a contradiction assume otherwise that there is an automorphism $\alpha$ of $G$ such that $a^2\alpha=b$. Now since $\alpha$ is an automorphism of $G$ so we have

$$\begin{aligned} b &= a^2\alpha \\ &= (a\alpha)^2 \end{aligned}$$

which is a contradiction because there is no element $c$ of $G$ such that $c^2 = b$.

We shall now find finite basis for the laws of pointed-groups (G,c) where G is the dihedral group i.e. $G=\langle a,b \mid a^3=b^2=1,\ bab^{-1}=a^{-1}\rangle$ of order 6, and where $c \in G$.

Now by Theorem 4.3.1 and Lemma 4.3.2 we only consider the pointed-groups (G,1), (G,a) and (G,b).

As a corollary to the Theorem 4.3.3 and corollary 4.2.6 we have:

**4.3.5 THEOREM**: The laws of (G,1) where G is a dihedral group of order 6 have the basis $\{y, x_1^6, [x_1^2, x_2^2]\}$.

Before continiuing with the pointed-groups (G,a) and (G,b) we need some facts about '*locally finite*' varieties of pointed-groups.

**DEFINITION**: A variety $\underline{\underline{W}}$ of pointed-groups is *locally finite* if every finitely generated pointed-group in $\underline{\underline{W}}$ is finite.

Thus by Theorem 2.3.6 we have:

**4.3.6 COROLLARY**: Every locally finite variety of pointed-groups is generated by its finite pointed-groups.

Now by Lemma 1.1.2 a pointed-group (G,c) is finitely generated if and only if G is finitely generated. Hence, it follows that a pointed-group variety is locally finite if and only if the finitely generated groups it contains as carriers are finite.

**DEFINITION**: A pointed-group (G,c) is called *menolithic* if its carrier G is monolithic group.

**4.3.7 LEMMA**: Let (A,a) be a pointed-group. Let M and N be normal sub groups of A such that $M \cap N = \{1\}$. Then (A,a) is isomorphic to a sub pointed-group of (A/M , aM) x (A/N , aN).

**Proof:** The mapping $\theta$: $A \to A/M \times A/N$ defined by

$$a_1\theta = (a_1M , a_1N), a_1 \in A$$

is easily checked to be a group monomorphism. But $a\theta=(aM , aN)$. Thus $\theta$ gives a pointed-group monomorphism from (A,a) to (A/M , aM) x (A/N , aN). Hence (A,a) is isomorphic to a sub pointed-group of (A/M , aM) x (A/N , aN) as required.

**4.3.8 LEMMA**: Let $\underline{\underline{V}}$ be a variety of pointed-groups which is generated by its finite pointed-groups. Then $\underline{\underline{V}}$ is generated by its monolithic pointed-groups.

**Proof:** Let $\underline{\underline{W}}$ be the sub variety of $\underline{\underline{V}}$ generated by the monolithic pointed-groups of $\underline{\underline{V}}$. To get a contradiction we assume that $\underline{\underline{W}} \subset \underline{\underline{V}}$. Then, since $\underline{\underline{V}}$ is generated by its finite pointed-groups and $\underline{\underline{W}}$ is a proper sub variety of $\underline{\underline{V}}$, so $\underline{\underline{W}}$ does not contain some finite-pointed groups of $\underline{\underline{V}}$.

Now suppose (G,c) is a finite pointed-group of $\underline{V}\backslash\underline{W}$ of smallest possible order. Then, since (G,c) $\notin \underline{W}$, so (G,c) is not monolithic. Thus by Lemma 4.3.7, (G,c) belongs to the variety generated by its proper factor pointed-groups. But these proper factor pointed-groups have order smaller than that of (G,c) and so are in $\underline{W}$. Thus (G,c) $\in \underline{W}$ which is a contradiction.

By Lemma 4.3.8 and corollary 4.3.6 we have:

**4.3.9** **<u>THEOREM</u>**: Every locally finite variety of pointed-groups is generated by its monolithic pointed-groups.

**4.3.10** **<u>THEOREM</u>**: The variety generated by any finite pointed-group is locally finite.

**Proof**: Let (A,a) be a finite pointed-group. Let (G,c) be a finitely generated pointed-group in the variety $\underline{V}$ generated by (A,a). We need to prove that (G,c) is finite.

Now by Theorem 2.2.11, (G,c) is isomorphic to a factor pointed-group of a relatively free pointed-group (F,t)($\underline{V}$) generated by a finite set S. Thus, it is enough to show that (F,t)($\underline{V}$) is finite. Now $\underline{V}$ is the variety defined by the set of laws of (A,a). Thus by Theorem 2.3.2, (F,t)($\underline{V}$) is isomorphic to a sub pointed-group of $(A,a)^{\Lambda}$, where $\Lambda$ is the set of functions from S to A. Now since S and A are finite so

$\Lambda$ is finite. Hence $(A,a)^\Lambda$ is finite. Thus $(F,t)(\underline{V})$ is finite as required.

**4.3.11** **<u>THEOREM</u>:** The laws of $(G,a)$ where $G = \langle a,b \mid a^3=b^2=1, ba\,b^{-1}=a^{-1}\rangle$ is a dihedral group of order 6, have the basis $\{y^3, x_1^6, [x_1^2, x_2^2]\}$.

**Proof:** Clearly $y^3$, $x_1^6$ and $[x_1^2, x_2^2]$ are laws of $(G,a)$. Let $\underline{V}$ be the variety of pointed-groups defined by $\{y^3, x_1^6, [x_1^2, x_2^2]\}$. To prove the theorem we want to prove that $\underline{V} = \operatorname{var}(G,a)$. To do this, we need to show that (i) $\operatorname{var}(G,a) \subseteq \underline{V}$ and (ii) $\underline{V} \subseteq \operatorname{var}(G,a)$.

But, clearly $\operatorname{var}(G,a) \subseteq \underline{V}$, because $(G,a)$ satisfies the laws of $\underline{V}$ i.e. $(G,a) \in \underline{V}$. To complete the proof, it only remains to show that $\underline{V} \subseteq \operatorname{var}(G,a)$. To do this, we first show that $\underline{V}$ is locally finite variety of pointed-groups. Now to prove that $\underline{V}$ is locally finite, we must show that every finitely generated pointed-group in $\underline{V}$ is finite. But, a pointed-group is finitely generated if and only if its carrier is finitely generated. Thus, it is enough to show that every finitely generated group in $\underline{V}$ is finite. But $\underline{V}$ has the laws $x_1^6$ and $[x_1^2, x_2^2]$ and we have proved in the last section that every finitely generated group in the variety defined by $\{x_1^6, [x_1^2, x_2^2]\}$ is finite. Hence it follows that $\underline{V}$ is locally finite variety of pointed-groups. Therefore, by Theorem 4.3.9, $\underline{V}$ is generated by its monolithic groups. Thus to prove that $\underline{V} \subseteq \operatorname{var}(G,c)$, it is enough to show that every monolithic pointed-group in $\underline{\underline{V}}$ belongs to the var(G,a). So let (C,c) be a monolithic pointed-group in $\underline{\underline{V}}$. Now since (C,c) is a monolithic pointed-group, so C is a monolithic group. But C has the laws $x_1^6$ and $[x_1^2, x_2^2]$. Hence Lemma 4.2.3, shows that either $C \cong C_2$ or $C \cong C_3$ or $C \cong G$. Also, (C,c) has the law $y^3$. Therefore, we have $c^3 = 1$. Hence to prove that $(C,c) \in \text{var}(G,a)$, we have to consider the following cases:

<u>Case (i)</u> $(C,c) \cong (G,a)$

<u>Case (ii)</u> $(C,c) \cong (G,a^2)$

<u>Case(iii)</u> $(C,c) \cong (G,1)$

<u>Case (iv)</u> C has order 2 and c=1

<u>Case (v)</u> C has order 3 and c has order 3

<u>Case (iv)</u> C has order 3 and c=1

Now in case (i) and (ii) we have $(C,c) \cong (G,a)$, because a is similar to $a^2$. Thus $(C,c) \in \text{var}(G,a)$ as required. In case (v) we have $(C,c) \cong (\langle a \rangle, a)$ i.e. (C,c) is isomorphic to a sub pointed group of (G,a). Again we have $(C,c) \in \text{var}(G,a)$ as required. Case (iv) and case (v) arise as sub pointed-groups of case (iii).

Hence to finish the proof we need only deal with case (iii). Now in case (iii) we want to prove that $(G,1) \in \text{var}(G,a)$. To do this, it is enough to find a factor pointed-group of a sub pointed-group of (G,a) x (G,a) which is isomorphic to (G,1).

Now let $A = \{(g,g) \mid g \in G\}$. Then it is easy to check that A is a sub group of GxG. Therefore, (A,(a,a)) is a sub pointed-group of $(G\text{x}G,(a,a))$. Also, $(G,a) \cong (A,(a,a))$ because the function $\theta:G \to A$ defined by $g\theta = (g,g)$ for all $g \in G$ gives a pointed-group isomorphism. Again, let $B=((1,1), (a,1), (a^2,1)\}$. Then clearly B is a normal sub group of GxG. Also, we have $A \cap B=\{(1,1)\}$. Therefore $(AB,(a,a))$ is a sub pointed group of $(G,a)\text{x}(G,a)$ of order 18. To complete the proof, we need to show that $(AB,(a,a)$ has a factor pointed-group isomorphic to $(G,1)$.

Now let $N = \langle (a,a) \rangle$. Then clearly N is a normal sub group of A. We show that N is a normal sub group of AB. To do this, it is enough to show that B normalizes N. But $(a,1)^{-1} (a,a)(a,1) = (a^{-1},1)((a,1) = (a,a)$. Hence B centralizes N. Thus B normalizes N. Therefore, N is a normal sub group of AB. Hence $(AB/N,N)$ is a factor pointed group of $(AB,(a,a))$ of order 6.

Now the elements $s=(a,1)N$ and $t=(b,b)N$ of AB/N do not commute, because

$$\begin{aligned} s^{-1}t^{-1}st &= (a,1)^{-1}\ (b,b)^{-1}\ (a,1)\ (b,b)N \\ &= (a^{-1},1)\ (b^{-1},b^{-1})\ (a,1)\ (b,b)N \\ &= (a,1)N \\ &\neq N. \end{aligned}$$

Hence AB/N is a non-abelian group of order 6. Thus AB/N is isomorphic to G. Thus $(G,1)$ is isomorphic to $(AB/N,N)$ i.e. $(G,1)$ is isomorphic to a factor pointed-group of a sub pointed-group of $(G,a)\ \text{x}\ (G,a)$ as required. Thus in all the cases above we have $(C,c) \in \text{var}(G,a)$. Hence $\underline{\underline{V}} = \text{var}(G,a)$. Thus the theorem follows.

**4.3.12** **THEOREM:** The laws of (G,b) where $G=\langle a,b \mid a^3=b^2=1, b^{-1}ab=a^{-1}\rangle$ is a diheadral group of order 6, have the basis $\{y^2\ ,\ (x_1^2)^y x_1^2\ ,\ x_1^6\ ,\ [x_1^2\ ,\ x_2^2]\}$

**Proof:** It is easy to check that $y^2$ and $(x_1^2)^y x_1^2$ are laws of (G,b). Now as in the proof of Theorem 4.3.11, it is enough to show that if (C,c) is a monolithic pointed-group in the variety $\underline{\underline{V}}$ defined by $\{y^2, (x_1^2)^y x_1^2, x_1^6, [x_1^2, x_2^2]\}$, then $(C,c) \in var(G,b)$. So let (C,c) be a monolithic pointed-group in $\underline{\underline{V}}$. Since (C,c) is a monolithic pointed-group, so C is a monolithic group. But C has the laws $x_1^6$ and $[x_1^2, x_2^2]$. Hence Lemma 4.2.3, shows that either $C \cong C_2$ or $C \cong C_3$ or $C \cong G$. Also (C,c) has the laws $Y^2$ and $(x_1^2)^y x_1^2$. Therefore, we have $c^2 = 1$ and if $C \cong C_3$ or $C \cong G$ then $c \neq 1$. Hence to prove that $(C,c) \in var(G,b)$ we have to consider the following cases:

Case (i) C,c) $\cong$ (G,b)

Case (ii) (C,c) $\cong$ (G,ab)

Case(iii) (C,c) $\cong$ $(G,a^2b)$

Case (iv) C has order 2 and c has order 2

Case (v) C has order 2 and c=1

Now in case (i), (ii) and (iii) we have $(C,c) \cong (G,b)$ because b is similar to ab and $a^2b$. Thus $(C,c) \in (G,b)$ as required. In case (iv) we have $(C,c) \cong (\langle b\rangle, b)$ i.e. (C,c) is isomorphic to a sub pointed-group of (G,b). Again, we have $(C,c) \in var(G,b)$ as required. Now to complete the proof it only remains to deal with case (v) i.e. we must prove that $(C,1) \in \mathrm{var}(G,b)$. To do this, it is enough to show that $(C,1)$ is isomorphic to a factor pointed-group of $(C,b) \times (C,b)$ where $C=\langle b\rangle$ because $(C,b) \in (G,b)$.

Let $N=\langle (b,b)\rangle$. Then clearly N is a normal sub group of CxC because CxC is abelian and N is a sub group of CxC. Therefore, $(C\mathrm{x}C/N,N)$ is a factor pointed-group of $(C,b)\mathrm{x}(C,b)$ of order 2. Clearly $(C,1) \cong (C\mathrm{x}C/N,N)$ as required. Hence in all of the cases above $(C,c) \in \mathrm{var}(G,b)$. Thus the theorem follows.

Below now we give tables showing the bases for the laws of pointed-groups $(D_n,c)$ where $D_n$ is a dihedral group of order 2n i.e. $D_n=\langle a,b \mid a^n=b^2 = 1,\ b^{-1}ab = a^{-1}\rangle$, where c ranges over representatives of the similarity classes in $D_n$. The similarity classes of $D_n$ are: $\{1\}$, $\{b,ab,\dots,a^{n-1}b\}$, and $\{a^i,a^{-i}\}$, for $1 \le i \le \frac{1}{2}(n-1)$, if n is odd; and $\{1]$, $\{b,a^2b,\dots,a^{n-2}b\}$, $\{ab,a^3b,\dots, a^{n-1}b\}$, $\{a^{n/2}\}$, and $\{a^i,a^{-i}\}$, for $1 \le i \le \frac{1}{2}(n-2)$, if n is even. Note that we are using Theorems 4.3.1, 4.3.3, 4.3.4 and lemma 4.3.2 for the completion of these tables.

## Table I

| S. No. | The variety generated by a pointed group | Basis for the laws of $(D_n,c)$ when n is odd and $1 \leq i \leq \frac{1}{2}(n-1)$ |
|---|---|---|
| 1. | $(D_n,1)$ | The word y together with the laws of $D_n$ |
| 2. | $(D_n,a^i) \equiv (D_n,a^{-i})$ where $1 \leq i \leq \frac{1}{2}(n-1)$ if n is odd. | The word $y^n$ together with the laws of $D_n$ |
| 3. | $(D_n,b) \equiv (D_n,ab) \equiv \ldots\ldots \equiv (D_n,a^{n-1}b)$ | The words $y^2$ and $(x_1^2)^y x_1^2$ together with the laws of $D_n$ |

## Table II

| S. No. | The variety generated by a pointed-group | Basis for the laws of $(D_n,c)$ when n is even and $1 \leq i \leq \frac{1}{2}(n-1)$ |
|---|---|---|
| 1. | $(D_n,1)$ | The laws of $D_n$ together with the word y. |
| 2. | $(D_n,b) \equiv (D_n,a^2b) \equiv \ldots\ldots \equiv (D_n,a^{n-2}b)$ | The laws of $D_n$ together with the words $y^2$ and $(x_1^2)^y x_1^2$ |
| 3. | $(D_n,ab) \equiv (D_n,a^3b) \equiv \ldots\ldots \equiv (D_n,a^{n-1}b)$ | The laws of $D_n$ together with words $y^2$ and $(x_1^2)^y x_1^2$ |
| 4. | $(D_n,a^i) \equiv (D_n,a^{-i})$ where $1 \leq i \leq \frac{1}{2}(n-2)$ if n is even | The laws of $D_n$ together with the word $y^n$ |
| 5. | $(D_n,a^m)$, $a^m \in Z(D_n)$ where $m=n/2$, when n is even | The laws of $D_n$ together with the words $y^2$ and $[y, x_1]$. |

Now in the proof of Theorems 4.3.11 and 4.3.12 a key fact was that the variety generated by the dihedral group of order 6 contains (upto isomorphism) only finitely many monolithic groups. We shall now prove the following theorem.

4.3.13 THEOREM

Let G be a finite group such that var G contains (upto isomorphism) only finitely many monolithic groups. Then var(G,c) is finitely based for all $c \in G$.

**Proof:** Let V be the set of all laws of G. Let $\underline{\underline{W}}$ denote the variety of pointed-groups defined by V. Because var G is locally finite (by 15.71 of [12]), it follows that $\underline{\underline{W}}$ is locally finite. Also, by the hypothesis on var G, $\underline{\underline{W}}$ has (upto isomorphsim) only finitely many monolithic pointed-groups. We know that var(G,c) $\subseteq \underline{\underline{W}}$ and also $\underline{\underline{W}}$ is finitely based, because var G is finitely based (by the main result of [13]).

Now choose a finitely based variety $\underline{\underline{V}}$ of pointed-groups such that var(G,c) $\subseteq \underline{\underline{V}} \subseteq \underline{\underline{W}}$ and also such that $\underline{\underline{V}}$ has (upto isomorphism) as fewer monolithic pointed-groups as possible.

We claim that var(G,c) = $\underline{\underline{V}}$. Then it follows that var(G,c) is finitely based. Suppose otherwise that var(G,c) $\subseteq \underline{\underline{V}}$. Since $\underline{\underline{V}}$ is locally finite, so it is generated by its monolithic pointed-groups. Hence there is a monolithic pointed-group (H,h) in $\underline{\underline{V}}$ such that (H,h) $\notin$ var(G,c). Thus there is a law w of (G,c) which is not a law of (H,h).

Let $\underline{\underline{V}}'$ be the variety defined by the set of laws of $\underline{\underline{V}}$ together with the law w. Then $\underline{\underline{V}}'$ is finitely based and also $\underline{\underline{V}}'$ has (upto isomorphism) fewer monolithic pointed-groups than $\underline{\underline{V}}$ and var(G,c) $\subseteq \underline{\underline{V}}' \subseteq \underline{\underline{W}}$ which is contradiction to the choice of $\underline{\underline{V}}$. Hence the theorem follows.

Theorem 4.3.13 may be applied in the case of "A-groups" as we shall now see:

**DEFINITION**: A finite group G is called an A-group if every sub group of G is abelian.

The following result is well known, it may be found in John Cassey's Ph.D thesis. We include a proof for the sake of completeness.

**4.3.14 THEOREM**: Suppose G is an A-group. Then var G contains (upto isomorphism) only finitely many monolithic groups.

**Proof**: Let $\underline{\underline{V}}$ be the variety generated by an A-group G. Then by (15.73) and (51.24) of [12] every finite group in $\underline{\underline{V}}$ is also an A-group. Thus by (1.66) of [9], every monolithic group in $\underline{\underline{V}}$ is critical. It follows by (52.11) of [12] that $\underline{\underline{V}}$ contains (upto isomorphism) only finitely many monolithic grous.

Thus as a corollary to the Theorem 4.3.13 we have:

**4.3.15** <u>**COROLLARY**</u>: suppose G is an A-group. Then var(G,c) is finitely based for all c $\in$ G.

Now let G be a finite group which is not an A-group. Then it is known that var G contains (upto isomorphis) infinitely many monolithic groups. Hence the method above cannot be used to prove that var(G,c) is finitely based for all c $\in$ G.

Suppose G is a finite group. Then one of the key facts in the proof of the theorem of Sheila Oates and M.B. Powell [13] that the var G is finitely based is that the var G contains (upto isomorphism) only finitely many critical groups. The obvious way to generalize this theorem to finite pointed-groups, is to consider critical pointed-groups defined as follows:

<u>**DEFINITION**</u>: A pointed group (G,c) is critical if (G,c) is finite and (G,c) does not belong to the variety generated by the proper sub pointed-groups and proper factor pointed-groups of (G,c).

**4.3.16** <u>**THEOREM**</u>: Let $\underline{\underline{V}}$ be a locally finite variety of pointed-groups. Then $\underline{\underline{V}}$ is generated by the cirtical pointed-groups it contains.

**Proof:** The proof is similar to the corresponding result for groups (see 51.41 of [121]).

Now in order to copy the methods used in the proof of Oates Powell Theorem we should need to know that if (G,c) is a finite pointed-group then var(G,c) contains (upto isomorphism) only finitely many critical pointed groups. Also, as we mentioned earlier that laws of a group G are included among the laws of a pointed-group (G,c) so it would seem plausible that a simple modification of the proof of the Oates-Powell Theorem would yield that every finite pointed-group has a finite basis for its laws. But, rather surprisingly this is not so. In fact, we have been able to find that there is a finite pointed-group (P,p) whose set of laws has no finite basis. Thus the generalization of the Oates-Powell Theorem to finite pointed-groups is false. Thereby we have been able to provide a counter example in Chapter V to the conjecture of Sheila Oates Macdonald for pointed-groups to be false.