

Chapter III

**SOME FINITELY BASED VARIETIES OF
POINTED - GROUPS**

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§ 3.1 NILPOTENT POINTED-GROUP VARIETIES

We call a pointed-group *nilpotent of class c*, if its carrier is a nilpotent group, of class c.

Here, in this section, we shall discuss varieties of pointed-groups in which every pointed-group is nilpotent.

By a *basis* of a variety we mean any set of laws defining that variety and by *finitely based* variety we mean a variety which has finite basis for its laws, that is, there is a finite set of laws of which every law is a consequence.

Our aim is to generalize the result proved by R.C.Lyndon in [10] that every nilpotent variety of groups is finitely based i.e. every nilpotent group variety has a finite basis for its laws. To do this, we shall need some notational definitions in the free pointed-group (X, γ) generated by x_1, x_2, x_3, \dots

Put $X_{(1)} = X$

and define for $c \geq 1$, $X_{(c+1)} = [X_{(c)}, X]$ i.e. the sub group of X generated by all the elements of the form $[a, b]$ where $a \in X_{(c)}$

and $b \in X$. It is easy to see that $X_{(c)}$ is a normal admissible sub group of (X, y) . Note also that $X_{(c)} = [x_1, x_2, \dots, x_c]-(X, y)$ for $c \geq 2$ (see Theorem 10.2.1 of [7]).

Moreover, for each non-negative integer c , we write \underline{N}_c to denote the variety of pointed-groups defined by the word $[x_1, x_2, \dots, x_{c+1}]$ whose closure is $X_{(c+1)}$. Thus \underline{N}_0 is the variety of all trivial pointed-groups and \underline{N}_1 is the variety of all abelian pointed-groups. We shall write \underline{A} as an alternate notation for \underline{N}_1 .

More generally \underline{N}_c is the variety of all pointed-groups which are nilpotent of class at most c .

A pointed-group variety \underline{V} is called *nilpotent*, if $\underline{V} \subseteq \underline{N}_c$ for some c . If $\underline{V} \subseteq \underline{N}_c$ but $\underline{V} \not\subseteq \underline{N}_{c+1}$, then we say that \underline{V} has class c .

We shall prove the following theorem, which is a generalization of a result proved by R.C.Lyndon in [10] that every nilpotent variety of groups is finitely based.

3.1.1. THEOREM: The laws of any nilpotent pointed-group variety are finitely based.

Before we prove the above Theorem 3.1.1 we need to establish some results.

Now for each positive integer n , write δ_n to denote the endomorphism of (X, Y) determined by $x_n \delta_n = 1$ and $x_i \delta_n = x_i$ for $i \neq n$. (Note that for existence and uniqueness we are using corollary 1.6.6). Thus the effect of δ_n on a word w is to delete all the occurrences of x_n from w . For example take a

word $w = w(y, x_1, x_2) = y^{-1} x_1 y x_2^{-1} x_1^{-1} x_2$, then

$$\begin{aligned} w\delta_1 &= (y^{-1} x_1 y x_2^{-1} x_1^{-1} x_2) \delta_1 \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{and } w\delta_2 &= (y^{-1} x_1 y x_2^{-1} x_1^{-1} x_2) \delta_2 \\ &= y^{-1} x_1 y x_1^{-1} \end{aligned}$$

Now for all positive integers i, m and n , it is easy to check that $x_i \delta_m \delta_n = x_i \delta_n \delta_m$ and $x_i \delta_n^2 = x_i \delta_n$. Thus by corollary 1.6.6, we have the relations $\delta_m \delta_n = \delta_n \delta_m$ and $\delta_n^2 = \delta_n$. Moreover, if w is any word we write $w(1-\delta_n)$ to denote $w(w\delta_n)^{-1}$. Thus we have

$$w = w(1-\delta_n)w\delta_n$$

Therefore, with this also we have,

$$\begin{aligned} w(1-\delta_m)\delta_n &= (w(w\delta_m)^{-1})\delta_n \\ &= (w\delta_n)(w\delta_m\delta_n)^{-1} \\ &= (w\delta_n)(w\delta_m\delta_n)^{-1}, \text{ as } \delta_m\delta_n = \delta_n\delta_m \end{aligned}$$

i.e. $w(1-\delta_m)\delta_n = w\delta_n(1-\delta_m)$ and

$$\begin{aligned} w(1-\delta_n)\delta_n &= (w(w\delta_n)^{-1})\delta_n \\ &= (w\delta_n)(w\delta_n\delta_n)^{-1} \\ &= (w\delta_n)(w\delta_n^2)^{-1} \\ &= (w\delta_n)(w\delta_n)^{-1}, \text{ as } \delta_n^2 = \delta_n \\ &= 1 \end{aligned}$$

Thus we have also the relations $w(1-\delta_n)\delta_n = 1$ and

$w(1-\delta_m)\delta_n = (w\delta_n)(1-\delta_m)$. Note that $w\delta_n$ and $w(1-\delta_n)$ are consequences of w .

3.1.2 THEOREM: Let T be a finite set of positive integers. Let w be a word in (X, y) such that $w\delta_t = 1$ for all $t \in T$. Then $w \in X_{(n)}$ where $|T| = n$.

Proof: See corollary 3.3.8 of [12].

3.1.3 THEOREM: Let w be a element of X_n . Then there are words w_T for each subset T of $\{1, 2, \dots, n\}$ such that:

- i. w is a product of the words w_T in some order.
- ii. Each w_T is a consequence of w .
- iii. Each $w_T\delta_t = 1$ for all $t \in T$ and also w_T is a word only in the variables y and x_t in X where $t \in T$.

Proof: Assume that w is a word in the variables y and x_1, x_2, \dots, x_n . Then consider,

$$w = w(1-\delta_1)w\delta_1 \tag{i}$$

But, $w(1-\delta_1) = w(1-\delta_1)(1-\delta_2)w(1-\delta_1)\delta_2$ and

$w\delta_1 = w\delta_1(1-\delta_2)w\delta_1\delta_2$ Thus (i) gives

$$w = w(1-\delta_1)(1-\delta_2)w(1-\delta_1)\delta_2w\delta_1(1-\delta_2)w\delta_1\delta_2$$

$$\text{i.e. } w = w(1-\delta_1)(1-\delta_2)w(1-\delta_1)\delta_2w(1-\delta_2)w\delta_1\delta_2$$

Continuing in this way we get

$$w = \prod_{T \subseteq \{1,2,3,\dots,n\}} w_T$$

where the product is taken over the sub sets T of $N = \{1,2,\dots,n\}$ and where

$$w_T = w \prod_{u \in T} (1-\delta_u) \prod_{t' \in N \setminus T} \delta_{t'}$$

Now apply δ_t where $t \in T$ to w_T above we have:

$$w_T \delta_t = (w \prod_{u \in T} (1-\delta_u) \prod_{t' \in N \setminus T} \delta_{t'}) \delta_t$$

But δ_t and $\delta_{t'}$ commute and also δ_t and $(1-\delta_u)$ commute and $w \dots (1-\delta_t) \delta_t \dots = 1$. Hence it follows that $w_T \delta_t = 1$. Since this holds for all $t \in T$, also w_T is a word only in the variables y and x_t where $t \in T$, because w_T has the form

$$v \prod_{t' \in N \setminus T} \delta_{t'}$$

where

$$v = w \prod_{u \in T} (1-\delta_u)$$

and hence does not contain $x_{t'}$ ($t' \notin T$). Thus the conditions (i) and (iii) are satisfied. Now condition (ii) is an easy consequence of the properties δ_n and $\{1-\delta_n\}$ describe above earlier. This completes the proof.

§ 3.2 FINITE BASIS FOR THE LAWS OF NILPOTENT POINTED-GROUP VARIETY (PROOF OF THEOREM 3.1.1)

Suppose \underline{V} is a nilpotent pointed-group variety of class at most c i.e. $\underline{V} \subseteq \underline{N}_c$. Let V be the normal admissible sub group of (X, y) which defines \underline{V} . Let $w \in V$ and suppose that $w = w(y, x_1, x_2, \dots, x_n)$. Then, by Theorem 3.1.3, w is equivalent to a set of word $\{w_T\}$ where each T is a finite set

of positive integers, with the properties given in the statement of the Theorem 3.1.3.

Now suppose $|T| \geq c+1$. Then, by Theorem 3.1.2, $w_T \in X_{(c+1)}$. Therefore, w_T is a consequence of the word $[x_1, x_2, \dots, x_{c+1}]$. But if $|T| < c+1$. Then (by the change of variables) w_T is equivalent to an element of $V \cap X_c = V_c$. Thus, in either case, w_T is a consequence of $V_{(c+1)}$. Hence it follows that w is a consequence of V_{c+1} . Thus V is equivalent to V_{c+1} . Now, by Lemma 2.5.1, $\underline{V}(X_{c+1}, Y) = \underline{V}(X, Y) \cap X_{c+1}$ i.e. $\underline{V}(X_{c+1}, Y) = \dot{V}_{c+1}$

Now since $\underline{V} \subseteq \underline{N}_c$, so we have $\underline{N}_c(X_{c+1}, Y) \subseteq \underline{V}(X_{c+1}, Y)$. Now, by corollary 2.2.9, $\underline{N}_c(X_{c+1}, Y) = K$ say, is normal in X_{c+1} . Thus $K \subseteq V_{c+1}$ and therefore, V_{c+1}/K is a sub group of X_{c+1}/K . But, $(X_{c+1}/K, YK) \in \underline{N}_c$ (by corollary 2.2.10). Therefore, X_{c+1}/K is a nilpotent group of class at most c . But X_{c+1} is a finitely generated group. Therefore, X_{c+1}/K is finitely generated nilpotent group.

Now Theorem 9.16 of [11] states that in a finitely generated nilpotent group every sub group is finitely generated. Hence it follows that V_{c+1}/K being a sub group of X_{c+1}/K is finitely generated. Suppose v_1K, v_2K, \dots, v_mK are generators of V_{c+1}/K i.e. $V_{c+1}/K = \langle v_1K, \dots, v_mK \rangle$. Then $V_{c+1} = \langle v_1, v_2, \dots, v_m, K \rangle$. For, if

$$V_0 = \langle v_1, v_2, \dots, v_m, K \rangle$$

Then clearly V_0 is contained in V_{c+1} . Let $v \in V_{c+1}$ then $vK \in V_{c+1}/K$. So vK can be written in the form

$$\begin{aligned}
 vK &= (v_{i_1} K)^{\xi_1} \dots (v_{i_n} K)^{\xi_n} \\
 &= (v_{i_1}^{\xi_1} \dots v_{i_n}^{\xi_n}) K
 \end{aligned}$$

Therefore, v has the form

$$v = v_{i_1}^{\xi_1} v_{i_2}^{\xi_2} \dots v_{i_n}^{\xi_n} k, \text{ where } k \in K$$

Hence $v \in V_0$ i.e. V_{c+1} is contained in V_0 . Therefore, $V_{c+1} = V_0$, as required. Thus V is equivalent to $K \cup \{v_1, \dots, v_m\}$. But, $K = \underline{N}_c(X_{c+1}, Y) \leq \underline{N}_c(X, Y)$ and $\underline{N}_c(X, Y)$ is the set of all consequences of $\{x_1, x_2, \dots, x_{c+1}\}$. Therefore, V is equivalent to the set

$$\{\{x_1, x_2, \dots, x_{c+1}\}, v_1, v_2, \dots, v_m\}$$

Thus \underline{V} is finitely based and the proof of Theorem 3.1.1 is complete.

In the notation of the Lemma 2.5.1 and Theorem 2.5.3 we have.

3.2.1 COROLLARY: For any nilpotent pointed-group variety \underline{V} of class c we have $\underline{V} = \underline{V}^{(c+1)}$ —, that is, a pointed-group (A, a) belongs to \underline{V} if and only if all its $(c+1)$ -generator sub pointed-groups belong to \underline{V} .

§ 3.3 METABELIAN POINTED-GROUP VARIETIES

Here in this section we describe some other pointed-

group varieties, all of whose sub-varieties are finitely based. Our aim is to generalize the result proved by D.E.Cohen in [3] that metabelian varieties of groups are finitely based.

We shall call a pointed group *metabelian* if its carrier is a metabelian group.

Now for any group G we write $G^{(1)}$ to denote the derived group $[G, G]$ of G and $G^{(2)}$ to denote the second derived group $[G^{(1)}, G^{(1)}]$. It is easy to check that $X^{(1)}$ and $X^{(2)}$ are the normal admissible sub groups of (X, y) which are closures of the words $[x_1, x_2]$ and $[[x_1, x_2], [x_3, x_4]]$ respectively in (X, y) . The varieties of pointed-groups defined by $[x_1, x_2]$ is the variety of all abelian pointed-groups (we call a pointed-group (G, a) abelian if G is abelian) and is denoted by \underline{A} while the variety of pointed-groups defined by $[[x_1, x_2], [x_3, x_4]]$ is the variety of all metabelian pointed-groups which we shall denote by $\underline{A} \underline{A}$. Clearly $\underline{A} \underline{A}$ is finitely based.

The main result of this section is the following:

3.3.1 THEOREM: Every sub variety of $\underline{A} \underline{A}$ is finitely based.

In order to prove Theorem 3.3.1 we need to prove some results which are generalization of the results in the varieties of groups.

3.3.2 **LEMMA:** Let \underline{V} be a finitely based variety of pointed-groups. Let V be the corresponding normal admissible sub group of (X,y) . Then every sub variety of \underline{V} is finitely based if and only if the maximal condition holds for the normal admissible sub groups of (X,y) which contain V .

Proof: Suppose that every sub variety of \underline{V} is finitely based and let $V_0 \subseteq V_1 \subseteq V_2 \subseteq V_3 \subseteq V_4 \subseteq \dots$, be an infinite ascending chain of normal admissible sub groups of (X,y) containing V . Let W be the union of the sub groups V_n , $n=1,2,\dots$. Then W is a normal admissible sub groups of (X,y) containing V . Hence W corresponds to a sub-variety \underline{W} of \underline{V} . Thus, by assumption, \underline{W} is finitely based. Thus W is the closure of some finite set of words say $\{w_1, w_2, \dots, w_k\}$. Now for each i , we have $w_i \in W$ where $1 \leq i \leq k$. Hence $w_i \in V_{n(i)}$ for some integer $n(i)$. Let $m = \max(n(1), n(2), \dots, n(k))$, then $\{w_1, w_2, \dots, w_k\} \subseteq V_m$. Now since V_m is the normal admissible sub group and W is the closure of $\{w_1, w_2, \dots, w_k\}$, it follows that $W \subseteq V_m$. But also since $V_m \subseteq W$, so we have $V_m = W$. Therefore, $V_m = V_{m+1} = V_{m+2} = \dots = W$. Thus the maximal condition holds for the normal admissible sub groups of (X,y) containing V .

Now to prove the converse, we assume that \underline{V} has a sub variety \underline{W} which is not finitely based and we will deduce that the maximal condition does not hold for the normal admissible sub groups of (X,y) containing V .

So, let W be the normal admissible sub group of (X, Y) corresponding to the sub variety \underline{W} of \underline{V} . Then $V \subseteq W$. But our assumption on \underline{W} means that W is not the closure of some finite set of words say $\{v_1, v_2, \dots, v_k\}$. Thus, by the assumption on \underline{W} , we must have $V \subset W$. Now let $V_0 = V$ and let w_1 be any element of $W \setminus V_0$. Let V_1 be the closure of $\{v_1, v_2, \dots, v_k, w_1\}$ in (X, Y) . Then $V_0 \subset V_1$ and so, by the assumption on \underline{W} , we have $V_1 \subset W$. Now let w_2 be any element of $W \setminus V_1$ and let V_2 be the closure of $\{v_1, v_2, \dots, v_k, w_1, w_2\}$ in (X, Y) . Then we have $V_0 \subset V_1 \subset V_2 \subset W$ (by similar arguments as above). Now continuing in this way, we get a strictly ascending infinite chain:

$$V_0 \subset V_1 \subset V_2 \subset \dots,$$

of normal admissible sub groups of (X, Y) containing V . Thus the maximal condition does not hold for such sub groups of (X, Y) as desired. Thus the Lemma follows:

Now since the normal admissible sub group of (X, Y) which defines the variety $\underline{A} \underline{A}$ is $X^{(2)}$ so to prove that every sub variety of $\underline{A} \underline{A}$ i.e. finitely based, it is enough (by the above Lemma 3.3.2) to prove that the maximal condition holds for the normal admissible sub groups of (X, Y) containing $X^{(2)}$.

3.3.3 LEMMA: Let V and W be normal admissible sub groups of (X, Y) such that $V \subseteq W$. Suppose that the maximal condition holds for the normal admissible sub groups of (X, Y) containing

W and suppose that the maximal condition holds for the normal admissible sub groups Z of (X, Y) which satisfy $V \subseteq Z \subseteq W$. Then the maximal condition holds for the normal admissible sub groups of (X, Y) containing V .

Proof: Let $V \subseteq Y_1 \subseteq Y_2 \subseteq \dots$ be an ascending chain of normal admissible sub groups of (X, Y) containing V . Then

$$Y_1 \cap W \subseteq Y_2 \cap W \subseteq Y_3 \cap W \subseteq \dots,$$

is an ascending chain of normal admissible sub groups of (X, Y) such that, for all i , $V \subseteq Y_i \cap W \subseteq W$ and $Y_1 W \subseteq Y_2 W \subseteq \dots$, is an ascending chain of normal admissible sub groups containing W . Thus, by the hypothesis of the Lemma, we have:

$$Y_m \cap W = Y_{m+1} \cap W = Y_{m+2} \cap W = \dots, \text{ for some } m$$

and $Y_n W = Y_{n+1} W = Y_{n+2} W = \dots, \text{ for some } n$

Now if $i \geq m$ and $i \geq n$, then we have:

$$Y_i \cap W = Y_{i+1} \cap W \text{ and } Y_i W = Y_{i+1} W$$

Now, since $Y_i \subseteq Y_{i+1}$, we have

$$\begin{aligned} Y_{i+1} &= Y_{i+1} \cap Y_{i+1} W \\ &= Y_{i+1} \cap Y_i W, \text{ as } Y_{i+1} W = Y_i W, \end{aligned}$$

i.e. $Y_{i+1} = Y_i(Y_{i+1} \cap W), \text{ as } Y_i \subseteq Y_{i+1}$

$$\begin{aligned} &= Y_i(Y_i \cap W), \text{ as } Y_{i+1} \cap W = Y_i \cap W \\ &= Y_i, \text{ as } Y_i \cap W \subseteq Y_i \end{aligned}$$

Thus $Y_1 = Y_{1+1} = Y_{1+2} = \dots$, where $1 = \max(m, n)$. This completes the proof of the Lemma.

Now by Theorem 3.1.1, we know that every sub variety

of the pointed-group variety \underline{A} is finitely based. Thus the maximal condition holds for the normal admissible sub groups of (X, y) containing $X^{(1)}$. Thus, to prove that every sub variety of \underline{A} is finitely based, it is sufficient (by Lemma 3.3.3) to prove that the maximal condition holds for the normal admissible sub groups Z of (X, y) which satisfy $X^{(2)} \subseteq Z \subseteq X^{(1)}$.

3.3.4 LEMMA: Let V and Z be normal admissible sub groups of (X, y) such that $V \subseteq Z$. Then Z/V is a normal admissible sub group of $(X/V, yV)$.

Proof: Let zV be any element of Z/V where $z \in Z$. Let θ be any endomorphism of $(X/V, yV)$. We want to show that $(zV)\theta \in Z/V$.

Now, for each i , suppose that $(x_i V)\theta = w_i V$ where $w_i \in (X, y)$. Since (X, y) is the free pointed-group on x_1, x_2, \dots , so there is an endomorphism α of (X, y) such that for all i , $x_i \alpha = w_i$.

Now, since Z is a normal admissible subgroup (X, Y) , we have $z\alpha \in Z$. Thus $(z\alpha)V \in Z/V$.

Now let β denote the natural homomorphism of (X, Y) onto $(X/V, yV)$. Then for each i , we have

$$\begin{aligned} (x_i)\beta\theta &= (x_i\beta)\theta \\ &= (x_i V)\theta \\ &= w_i V, \end{aligned}$$

While

$$\begin{aligned}(x_i)\alpha\beta &= (x_i\alpha)\beta \\ &= w_i\beta \\ &= w_iV\end{aligned}$$

Thus $\beta\theta = \alpha\beta$ and, therefore, we have

$$\begin{aligned}(z\alpha)V &= (z\alpha)\beta \\ &= (z)\alpha\beta \\ &= (z)\beta\theta \\ &= (z\beta)\theta \\ &= (zV)\theta\end{aligned}$$

Since we have shown already that $(z\alpha)V$ belongs to Z/V , so we have $(zV)\theta \in Z/V$, as desired. Thus the Lemma follows.

3.3.5 LEMMA: Let V and W be the normal admissible sub groups of (X, y) such that $V \subseteq W$. Suppose $(X/V, yV)$ satisfies the maximal condition on the normal admissible sub groups contained in W/V . Then (X, y) satisfies the maximal condition for the normal admissible sub groups Z such that $V \subseteq Z \subseteq W$.

Proof: Let $Z_1 \subseteq Z_2 \subseteq Z_3 \subseteq \dots$, be an ascending chain of normal admissible sub groups of (X, y) such that for all i , $V \subseteq Z_i \subseteq W$. Then, by Lemma 3.3.4, we have

$$Z_1/V \subseteq Z_2/V \subseteq \dots,$$

is an ascending chain of normal admissible sub groups of $(X/V, yV)$ contained in W/V i.e. for all i , $Z_i/V \subseteq W/V$. But, by hypothesis, this chain is ultimately stationary. Hence the

chain $Z_1 \subseteq Z_2 \subseteq \dots$, is ultimately stationary. Thus the Lemma follows.

Now let $G = X/X^{(2)}$. Then $G^{(1)} = X^{(1)}/X^{(2)}$. Thus to prove that every sub variety of $\underline{A} \underline{A}$ is finitely based, it is enough to show (by Lemma 3.3.5) that the maximal condition is satisfied for the normal admissible sub groups of $(G, yX^{(2)})$ contained in $G^{(1)}$.

Notice that G is a relatively free group with free generators $yX^{(2)}, x_1X^{(2)}, \dots$. We shall change the notation for the free generators of G and shall write z_1 to denote $yX^{(2)}$ and z_{i+1} to denote $x_iX^{(2)}$ for all $i \geq 1$. Hence (G, z_1) is a relatively free pointed-group generated by z_2, z_3, z_4, \dots .

3.3.6 LEMMA: The maximal condition is satisfied for normal admissible sub groups of (G, z_1) , contained in $G^{(1)}$, if every normal admissible sub groups of (G, z_1) , contained in $G^{(1)}$, is the closure of some finite sub set.

Proof: The proof is similar to the proof of Lemma 3.3.2.

§ 3.4 FINITE BASIS FOR THE LAWS OF METABELIAN POINTED-GROUPS VARIETY (PROOF OF THEOREM 3.3.1)

DEFINITION: Let W be a set with a binary relation \preceq . We say

that \preceq is a quasi-order if:

- i. $s \preceq s$ for all $s \in W$.
- ii. $s_1 \preceq s_2$ and $s_2 \preceq s_3 \Rightarrow s_1 \preceq s_3$ for all $s_1, s_2, s_3 \in W$.

We say that \preceq is a *well-quasi-order* if it satisfies the following additional condition:

- iii. For every subset S of W there is a finite subset S_0 of S such that, for each element s of S , there is an element s_0 of S_0 such that $s_0 \preceq s$.

DEFINITION: Let W be a set with a binary relation \leq . We say that \leq is a *linear-order* if.

- i. $w \leq w$ for all $w \in W$,
- ii. $w_1 \leq w_2$ and $w_2 \leq w_3 \Rightarrow w_1 \leq w_3$ for all $w_1, w_2, w_3 \in W$,
- iii. $w_1 \leq w_2$ and $w_2 \leq w_1 \Rightarrow w_1 = w_2$ for all $w_1, w_2 \in W$,
- iv. For $w_1, w_2 \in W \Rightarrow$ either $w_1 \leq w_2$ or $w_2 \leq w_1$.

DEFINITION: A linear-order \leq on a set W is said to be a *well-order* if every non empty subset S of W contains a least element i.e., an element s_0 such that $s_0 \leq s$ for all $s \in S$.

DEFINITION: Let g and h be elements of $G^{(1)}$. We say g is a *consequence* of h if g belongs to the closure of h in $(G^{(1)}, z_1)$. We say that g is *equivalent* to h , written as $g \sim h$, if g is a consequence of h and h is a consequence of

$g \sim h$; that is, if $\langle g \rangle$ and $\langle h \rangle$ have the same closure.

The relation \sim defined above is an equivalence relation on $G^{(1)}$.

3.4.1 LEMMA: Suppose that W is a set with a well-order \leq and well-quasi-order \preceq . Suppose that χ is a function from $G^{(1)} \setminus \{1\}$ to W such that, whenever g and h are elements of $G^{(1)} \setminus \{1\}$ satisfying $\chi(g) \preceq \chi(h)$, then there is a consequence g_1 of g and an element h_1 equivalent to h such that either $h_1 = g_1$ or $\chi(h_1^{-1} g_1) < \chi(h)$. Then every normal admissible sub group of (G, z_1) contained in $G^{(1)}$, is the closure of a finite set of elements of (G, z_1) .

Proof: Suppose that u is a normal admissible sub group of (G, z_1) contained in $G^{(1)}$. Let $S = \{\chi(u) \mid u \in U \setminus \{1\}\}$. Now since \preceq is a well-quasi-order on W , so there is a finite subset S_0 of S such that, for every s of S , there is an element s_0 of S_0 such that $s_0 \preceq s$.

Let A be a finite subset of $U \setminus \{1\}$ so chosen that $S_0 = \{\chi(u) \mid u \in A\}$. Let U_0 be the closure of A in (G, z_1) . Thus $U_0 \subseteq U$. The proof will be complete if we show that $U_0 = U$.

Now suppose otherwise and let T be the non-empty set $T = \{\chi(u) \mid u \in U \setminus U_0\}$. Now, since \leq is a well-order on W ,

so there is a least element of T i.e., an element h of $U \setminus U_0$ such that $\chi(h) \leq \chi(u)$ for all $u \in U \setminus U_0$. Since $\chi(h) \in S$, so there is an element g of A such that $\chi(g) \leq \chi(h)$. Thus, by the hypothesis of the Lemma, there is a consequence g_1 of g and an element h_1 equivalent to h such that either $g_1 = h_1$ or $\chi(h_1 g_1^{-1}) < \chi(h)$. Since g_1 is a consequence of g , we have $g_1 \in U_0$. Also, since h_1 is equivalent to h , we have $h_1 \in U \setminus U_0$. Hence $h_1 \neq g_1$ and so we have $\chi(h_1 g_1^{-1}) < \chi(h)$. But $h_1 g_1^{-1} \in U \setminus U_0$. Hence $\chi(h) \leq \chi(h_1 g_1^{-1})$ which is a contradiction. Thus the Lemma follows.

Now to prove that every sub variety of \underline{A} is finitely based, it is enough to show (by Lemma 3.4.1) that there is a set W with a well-order \leq , a well-quasi-order \preceq and a function $\chi: G^{(1)} \setminus \{1\} \rightarrow W$ satisfying the condition of Lemma 3.4.1

To do this, first it will be necessary to describe a suitable set W with a well-order \leq and well-quasi-order \preceq .

Let J denote the set of non-negative integers. Let $V(J)$ denote the set of infinite sequences $\alpha = (\alpha_i) = (\alpha_1, \alpha_2, \alpha_3, \dots)$ of non negative integers in which $\alpha_i = 0$ for all but finitely many values of i . Let $V_3(J)$ denote the set $P \times P \times P \times V(J)$ where P denotes the set of positive integers. Let Φ denote the set of all functions $\phi: J \rightarrow J$ such that $0\phi = 0$ and $i\phi < j\phi$ whenever $i < j$. Now we define a

relation \preceq on $V_3(J)$ as follows:

We write $(r, s, t, (\alpha_i)) \preceq (r', s', t', (\alpha'_i))$ if there exists an element $\phi \in \Phi$ such that $r\phi = r'$, $s\phi = s'$, $t\phi = t'$ and $\alpha_i \leq \alpha'_{i\phi}$ for all $r, s, t, r', s', t', i, i\phi \in P$ and $\alpha_i, \alpha'_{i\phi} \in J$.

3.4.2 LEMMA: \preceq is a well-quasi-order on $V_3(J)$.

Proof: Clearly \preceq is a well-quasi-order on J . Hence the Lemma is a special case of the Lemma 3.2 of [2].

Next consider the subset V of $V_3(J)$ consisting of those elements $(r, s, t, (\alpha_i))$ in which $t=1$. For brevity, we write $(r, s, 1, (\alpha_i))$ simply as $(r, s, (\alpha_i))$. Now since $V_3(J)$ has a well-quasi-order \preceq , it follows that V has a well-quasi-order \preceq . In the abbreviated notation, we have:

$$(r, s, (\alpha_i)) \preceq (r', s', (\alpha'_i))$$

if and only if there exists $\phi \in \Phi$ such that $1\phi = 1$, $r\phi = r'$, $s\phi = s'$ and $\alpha_i \leq \alpha'_{i\phi}$ for all $\alpha_i, \alpha'_{i\phi} \in J$ and $r, r', s, s', i, i\phi \in P$.

Now suppose W is the set $V \times P$. We define a relation \preceq on W as follows:

We write $(v, p) \preceq (v', p')$ if $v \leq v'$ and $p \leq p'$.

3.4.3 LEMMA: \preceq is a well-quasi-order on W .

Proof: Since \leq is a well-quasi-order on P and \preceq is a well-quasi-order on V , the Lemma is a special case of the Theorem 2.3 of [8].

We have defined above, a set W with a well-quasi-order. The next task is to define a suitable well-order on W . To do this, first we need to define a well-order \leq on $V(J)$ as follows:

We write $(\alpha_i) \leq (\acute{\alpha}_i)$ if either $(\alpha_i) = (\acute{\alpha}_i)$ or there exists j such that $\alpha_j < \acute{\alpha}_j$, but $\alpha_i = \acute{\alpha}_i$ for all $i > j$. It is easy to check that this is a well-order. Now we define a well-order \leq on V as follows:

We write $(r, s, (\alpha_i)) \leq (r', s', (\acute{\alpha}_i))$ if either $r < r'$, $s < s'$ or $r=r'$, $s = s'$ and $(\alpha_i) \leq (\acute{\alpha}_i)$. It is easy to check that \leq is well-order on V .

Now we are ready to define a well-order \leq on W as follows:

We write $(v, p) \leq (v', p')$, if either $v < v'$ or $v=v'$ and $p \leq p'$. It is straightforward to check that \leq is a well-order on W .

3.4.4 **LEMMA:** Suppose that $(r, s, (\alpha_i))$ and $(\hat{r}, \hat{s}, (\hat{\alpha}_i))$ are elements of V such that $(\hat{r}, \hat{s}, (\hat{\alpha}_i)) < (r, s, (\alpha_i))$. Let ϕ be any element of Φ . Then $(\hat{r}\phi, \hat{s}\phi, (\hat{\beta}_i)) < (r\phi, s\phi, (\beta_i))$ where $\hat{\beta}_{i\phi} = \hat{\alpha}_i$ and $\beta_{i\phi} = \alpha_i$ for all i , while $\hat{\beta}_j = 0$ and $\beta_j = 0$ if j is not in the range of ϕ .

Proof: The proof is straight forward.

3.4.5 **LEMMA:** Suppose that $(r, s, (\alpha_i))$ and $(\hat{r}, \hat{s}, (\hat{\alpha}_i))$ are the elements of V such that $(\hat{r}, \hat{s}, (\hat{\alpha}_i)) < (r, s, (\alpha_i))$. Let $(\gamma_i) \in V(J)$. Then $(\hat{r}, \hat{s}, (\hat{\alpha}_i + \gamma_i)) < (r, s, (\alpha_i + \gamma_i))$.

Proof: The Proof is straightforward.

Now the next task is to define a suitable function $\chi: G^{(1)} \setminus \{1\} \rightarrow W$ so that this function χ has the properties required in the Lemma 3.4.1.

Let N be the normal closure in G of the set of all commutators $[z_r, z_s]$ where $r, s \in P$. then G/N is generated by the elements $z_i N$, $z_i \in G$, and these elements commute. Hence G/N is abelian. Thus $G^{(1)} \subseteq N$. But, $[z_r, z_s] \in G^{(1)}$ for all $r, s \in P$. Hence $N \subseteq G^{(1)}$. Thus $G^{(1)} = N$. Therefore, $G^{(1)}$ is generated by the conjugates of the elements $[z_r, z_s]$.

Now let w be any element of G . Then we can write w in the form:

$$w = h z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}$$

where $h \in G^{(1)}$ and the α_i are integers. Thus

$$[z_r, z_s]^w = [z_r, z_s]^{h z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}}$$

i.e.
$$[z_r, z_s]^w = [z_r, z_s]^{z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}}$$

(because $G^{(1)}$ is abelian and so $[z_r, z_s]^h = [z_r, z_s]$)

Consequently every element of $G^{(1)}$ is a product of elements of the form

$$[z_r, z_s]^{z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}}$$

and their inverses. Moreover, $G^{(1)}$ is abelian, so these elements commute, since

$$\begin{aligned} (g z_1^{\alpha_1} z_2^{\alpha_2}) z_1^{\beta_1} z_2^{\beta_2} &= [(z_1^{\alpha_1} z_2^{\alpha_2}) (g) (z_1^{\alpha_1} z_2^{\alpha_2})^{-1}] z_1^{\beta_1} z_2^{\beta_2} \\ &= (z_1^{\beta_1} z_2^{\beta_2}) (z_1^{\beta_1} z_2^{\alpha_2}) (g) (z_1^{\alpha_1} z_2^{\alpha_2})^{-1} (z_1^{\beta_1} z_2^{\beta_2})^{-1} \\ &= z_1^{\alpha_1 + \beta_1} [z_1^{-\alpha_1}, z_2^{\beta_2}] (z_2^{\alpha_2 + \beta_2} g z_2^{-\alpha_2 - \beta_2}) [z_2^{\beta_2}, z_1^{-\alpha_1}] z_1^{-\alpha_1 - \beta_1} \\ &= z_1^{\alpha_1 + \beta_1} (z_2^{\alpha_2 + \beta_2} g z_2^{-\alpha_2 - \beta_2}) [z_1^{-\alpha_1}, z_2^{\beta_2}] [z_2^{\beta_2}, z_1^{-\alpha_1}] z_1^{-\alpha_1 - \beta_1} \\ &= z_1^{\alpha_1 + \beta_1} (z_2^{\alpha_2 + \beta_2} g z_2^{-\alpha_2 - \beta_2}) z_1^{-\alpha_1 - \beta_1} \\ &= (z_1^{\alpha_1 + \beta_1} z_2^{\alpha_2 + \beta_2}) (g) (z_1^{\alpha_1 + \beta_1} z_2^{\alpha_2 + \beta_2})^{-1} \\ &= (g) z_1^{\alpha_1 + \beta_1} z_2^{\alpha_2 + \beta_2} \end{aligned}$$

Thus,
$$([z_r, z_s]^{z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n} z_1^{\beta_1} z_2^{\beta_2} \dots z_n^{\beta_n}})$$

$$= [z_r, z_s]^{z_1^{\alpha_1+\beta_1} z_2^{\alpha_2+\beta_2} \dots z_n^{\alpha_n+\beta_n}}$$

Now if we choose $\beta_1, \beta_2, \dots, \beta_n$ to be large enough positive integers, then $\alpha_1 + \beta_1, \alpha_2 + \beta_2, \dots, \alpha_n + \beta_n$ will be non-negative.

Consequently, if g is any element of $G^{(1)}$ then there is a conjugate $g^{z_1^{\beta_1} z_2^{\beta_2} \dots z_n^{\beta_n}}$ of g such that g is a product of elements of the form

$$[z_r, z_s]^{z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}}$$

and their inverses where the α_i are non-negative integers. We call such an element of $G^{(1)}$ a *positive element*. Hence every element of $G^{(1)}$ is equivalent to a positive element. Now consider the element;

$$[z_r, z_s]^{z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}}$$

where the α_i are non-negative integers. With this element we can associate an element of V namely $(r, s, (\alpha_i))$, where we take $\alpha_i = 0$ for $i > n$.

Now suppose g_1 is a positive element of $G^{(1)} \setminus \{1\}$.

Since the elements $[z_r, s_s] z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}$ commute, so g_1 can be written as a product of powers of distinct positive elements $[z_r, z_s] z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n}$. We choose the power say

$$([z_r, z_s] z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n})^p$$

where $p \neq 0$ in which $(r, s, (\alpha_i))$ is the largest in the well order \leq on V defined above. We call this the *leading term* of g_1 . Now if p is negative we consider g_1^{-1} instead of g_1 where g_1^{-1} is equivalent to g_1 and if p is positive we may associate with g_1 , the element $(r, s, (\alpha_i), p)$ of W .

Now suppose $g \in G^{(1)} \setminus \{1\}$. As g_1 varies over the set of positive elements equivalent to g , the leading terms of the elements g_1 give a set of elements $(r, s, (\alpha_i), p)$ of W .

We define $\chi: G^{(1)} \setminus \{1\} \rightarrow W$ such that $\chi(g)$ is the least element $(r, s, (\alpha_i), p)$ in this set of elements of W (using the well-order \leq we have defined on W). To complete the proof, it remains to show that if g and h are elements of $G^{(1)} \setminus \{1\}$ satisfying $\chi(g) \leq \chi(h)$, then there is a consequence g_1 of g and an element h_1 equivalent to h such that either $h_1 = g_1$ or $\chi(h_1 g_1^{-1}) < \chi(h)$.

Now suppose then that $g, h \in G^{(1)} \setminus \{1\}$ such that $\chi(g) \leq \chi(h)$.

Let $\chi(g) = (r, s, (\alpha_i), p)$ and
 $\chi(h) = (r', s', (\alpha'_i), p')$

Now by the definition of χ , there is a positive element g' equivalent to g with leading term

$$((z_r, z_s) z_1^{\alpha_1} z_2^{\alpha_2} \dots)^p.$$

Similarly there is a positive element h equivalent to h with leading term

$$((z_{r'}, z_{s'}) z_1^{\alpha'_1} z_2^{\alpha'_2} \dots)^{p'}.$$

Now $\chi(g) \preceq \chi(h)$. Therefore, by the definition of \preceq we have $p \leq p'$ and there is an element ϕ of Φ such that $1\phi = 1$, $r\phi = r'$, $s\phi = s'$ and $\alpha_i\phi \leq \alpha'_i$ for all i . Now by Theorem 2.2.12 there is an endomorphism θ of (G, z_1) such that $z_i\theta = z_i\phi$ for all $i \geq 2$. But, since $1\phi = 1$, we have $z_1\theta = z_1\phi$ for all $i \geq 1$.

Now clearly $g'\theta$ is a positive element of $G^{(1)} \setminus \{1\}$ and by Lemma 3.4.4 the leading term of $g'\theta$ is

$$((z_{r\phi}, z_{s\phi}) z_1^{\alpha_1\phi} z_2^{\alpha_2\phi} \dots)^p$$

i.e. $((z_{r'}, z_{s'}) z_1^{\beta_1} z_2^{\beta_2} \dots)^p$

where $\beta_i \leq \alpha'_i$ for all i .

Now let $g_1 = (g'\theta) z_1^{\alpha_1 - \beta_1} z_2^{\alpha_2 - \beta_2} \dots$. Then clearly g_1 is a positive element and by Lemma 3.4.5 the leading term of g_1 is

$$((z_{r'}, z_{s'}) z_1^{\alpha_1} z_2^{\alpha_2} \dots)^p$$

Clearly g_1 is a consequence of g and we have $p \leq p'$. Now if $p < p'$, then the leading term of $h_1 \bar{g}_1^{-1}$ is

$$([z_{r'}, z_{s'}]^{z_1^{\alpha'_1} z_2^{\alpha'_2} \dots})^{(p'-p)}$$

while if $p = p'$ then either $h_1 = g_1$ or else $h_1 \bar{g}_1^{-1}$ has leading term which is a power of an element

$$([z_{r''}, z_{s''}]^{z_1^{\alpha''_1} z_2^{\alpha''_2} \dots},$$

in which $(r'', s'', (\alpha''_i)) < (r', s', (\alpha'_i))$. Thus if $h_1 \neq g_1$ then $\chi(h_1 g^{-1}) < \chi(h)$ as required.

Now, by lemma 3.4.1, every normal admissible subgroup of (G, z_1) , contained in $G^{(1)}$, is the closure of a finite set of elements of (G, z_1) . Hence every sub variety of $\underline{\mathbb{A}} \underline{\mathbb{A}}$ is finitely based. This proves Theorem 3.3.1.

3.4.6 COROLLARY: The metabelian variety $\underline{\mathbb{A}} \underline{\mathbb{A}}$ of all metabelian pointed-groups is finitely based.