

Chapter II

LAWS IN A POINTED - GROUP

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§ 2.1 WORDS AND LAWS

It will be convenient to have some notation for a particular free pointed-group (X, y) of countable rank where X will denote the free group on a generating set consisting of symbols y, x_1, x_2, \dots . The symbols will be called variables. Any element of X will be called a *word*. Each and every word w can be written in the standard form given by:

$$w = w(y, x_1, x_2, \dots, x_n) \quad (i)$$

because it contains only finitely many variables.

Note that we use the notation (i) for every word w ; in spite of the fact that there may be words in which the particular variable y does not actually occur.

Suppose that (G, c) is a pointed-group and suppose α is a homomorphism from (X, y) to (G, c) , so that $y\alpha = c$. Let w be a word in X . Then the image $w\alpha$ of w under α is called a *value* of w in (G, c) . Of course, the value is an element of G .

Suppose

$$w = w(y, x_1, x_2, \dots, x_n)$$

and $x_1\alpha = g_1, x_2\alpha = g_2, \dots, x_n\alpha = g_n$, then we write:

$$w\alpha = w(c, g_1, g_2, \dots, g_n)$$

i.e. the value $w\alpha$ is obtained from w by substituting arbitrary elements of G for the variables x_1, x_2, \dots, x_n and the particular element c of G for the variable y .

Conversely, if g_1, g_2, \dots, g_n are arbitrary elements of G , then the element $w(c, g_1, g_2, \dots, g_n)$ of G arising by substitution in this way is a value of w in (G, c) , because by Theorem 1.6.2, there is a homomorphism α from (X, y) to (G, c) such that $x_1\alpha = g_1, x_2\alpha = g_2, \dots, x_n\alpha = g_n$.

Now let w be a word. Let (G, c) be a pointed-group. Write $w(G, c)$ to denote the sub group of G generated by all value of w in (G, c) .

More generally, if W is a set of words, then we define $W(G, c)$ to be the sub group of G generated by all values in (G, c) of all the elements of W . We call $W(G, c)$ the *word sub group* of (G, c) corresponding to W .

EXAMPLES: (i) Let $G = \langle a, b \mid a^3 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$. Let $W = \{Y^3\}$. Take the element b in G and consider the pointed-group (G, b) . Now the only value of Y^3 in (G, b) is the element $b^3 = b$. Therefore, $W(G, b) = \langle b \rangle$.

(ii) Let $G = \langle a, b \mid a^2 = b^2 = 1, [a, b] = 1 \rangle$. Take the word $w = [y, x_1]$ and consider the pointed-group (G, a) . Now the values

of w in (G, a) are the elements of G of the form $[a, g_1]$, $g_1 \in G$. Now since G is abelian we have $[a, g_1] = 1$ for all $g_1 \in G$. Therefore, 1 is the only value of w in (G, a) . Thus $w(G, a) = \{1\}$.

Suppose $\alpha : (G, c) \rightarrow (H, d)$ is a homomorphism and let $w = w(x_1, x_2, \dots, x_n)$ be a word in (X, Y) . Then for all $g_1, g_2, \dots, g_n \in G$ we have,

$$w(c, g_1, g_2, \dots, g_n)\alpha = w(c\alpha, g_1\alpha, \dots, g_n\alpha),$$

Therefore, the set of all values of w in $(G, c)\alpha$ is the image under α of the set of all values of w in (G, c) .

More generally, if W is a set of words then the set of all values in (G, c) of all the elements of W is the image under α of the set of all values in (G, c) of all the elements of W . Thus we have:

2.1.1 LEMMA: Let $\alpha : (G, c) \rightarrow (H, d)$ be a homomorphism and let W be a set of words. Then $W(G, c)\alpha = W(G\alpha, d)$

2.1.2 COROLLARY: Let (G, c) be a pointed-group. Let N be a normal sub group of G . Let W be a set of words. Then $W(G/N, cN) = W(G, c)N/N$.

Proof: Let $\alpha : (G, c) \rightarrow (G/N, cN)$ be the natural epimorphism. Then by the above Lemma 2.1.1 we have

$$W(G, c)\alpha = W(G/N, cN)$$

Thus $W(G/N, cN) = W(G, c)N/N$.

2.1.3 LEMMA: Every word sub group $W(G,c)$ of a pointed-group (G,c) corresponding to a set of words W is an admissible sub group of (G,c) .

Proof: Let α be an endomorphism of (G,c) . Then by Lemma 2.1.1, we have $W(G,c)\alpha = W(G\alpha,c)$. Thus $W(G,c)\alpha \leq W(G,c)$ for all $\alpha \in \text{End}(G,c)$. Thus the Lemma follows:

Suppose W is a set of words. Then by definition, $W(X,y)$ is generated by all the elements of the form $w\alpha$ where $w \in W$ and α is an endomorphism of (X,y) . Therefore, by Theorem 1.5.2 $W(X,y)$ is the admissible closure of W in (X,y) i.e. $W(X,y) = \bar{W}$. Thus we have:

2.1.4 LEMMA: Let W be a set of words and let \bar{W} be the admissible closure of W in (X,y) . Let (G,c) be a pointed-group. Then $W(G,c) = \bar{W}(G,c)$

Proof: Clearly $W(G,c) \subseteq \bar{W}(G,c)$. Thus it only remains to show that $\bar{W}(G,c) \subseteq W(G,c)$. To do this, it is enough to show that $W(G,c)$ contains every value $v\alpha$ where $v \in \bar{W}$ and α is a homomorphism from (X,y) to (G,c) . Thus it is enough to show that $\bar{W}\alpha \subseteq W(G,c)$. But \bar{W} is generated by elements of the form $w\beta$ where $w \in W$ and β is an endomorphism of (X,y) . Thus it is enough to show that $w\beta\alpha \in W(G,c)$. But this holds, because $\beta\alpha$ is a homomorphism from (X,y) to (G,c) : Thus $w\beta\alpha$ is a value of $w \in W$ in (G,c) . Thus the Lemma follows.

2.1.5 COROLLARY: Let W_1 and W_2 be wet of words. Then $W_1(G,c) = W_2(G,c)$ for every pointed-group (G,c) if and only if $\bar{W}_1 = \bar{W}_2$.

Proof: First suppose that $\bar{W}_1 = \bar{W}_2$. Then by the above Lemma 2.1.4 we have $W_1(G,c) = \bar{W}_1(G,c)$ and $W_2(G,c) = \bar{W}_2(G,c)$. Thus $W_1(G,c) = W_2(G,c)$.

Conversely, suppose that $W_1(G,c) = W_2(G,c)$ for every pointed-group (G,c) . Then $W_1(X,Y) = W_2(X,Y)$. But $W_1(X,Y) = \bar{W}_1$ and $W_2(X,Y) = \bar{W}_2$. Thus $\bar{W}_1 = \bar{W}_2$.

DEFINITION: Let $w = w(y, x_1, x_2, \dots, x_n)$ be a word and let (G,c) be a pointed group. The word w is said to be a law of (G,c) , if every value of w in (G,c) is trivial i.e. if $w(c, g_1, g_2, \dots, g_n) = 1$ for all $g_1, g_2, \dots, g_n \in G$ i.e. if $w\alpha = 1$ for every homomorphism α from (X,y) to (G,c) . Thus w is a law of (G,c) if and only if $w(G,c) = \{1\}$.

EXAMPLE: Let $G = \langle a, b \mid a^3 = 1 = b^2, b^{-1}ab = a^{-1} \rangle$. Consider the pointed-group (G,a) . Take the word $w(y, x_1) = [y, x_1]^2$. Let g_1 be any element of G . Then $[a, g_1]^2 = 1$. Therefore $w(y, x_1)$ is a law of (G,a) i.e. $[y, x_1]^2$ is a law of (G,a) . By contrast if $w(y, x_1, x_2) = [x_2, x_1]^2$ then $w(y, x_1, x_2)$ is not a law of (G,a) , as can be seen by substituting b for x_2 and a for x_1 .

Note that the laws of (G, c) includes as a special case the ordinary laws of G . For, if $w = w(x_1, x_2, \dots, x_n)$ is a law of G , then we write the word $w = w(x_1, x_2, \dots, x_n)$ in the standard form

$$w = w(y, x_1, x_2, \dots, x_n)$$

where the variable y does not actually occur. Therefore, the idea of a law of (G, c) generalizes the idea of a Law of G .

2.1.6 LEMMA: i. If w is a law of a pointed-group (G, c) , then w is a law of every sub pointed-group and factor pointed-group of (G, c) .

ii. Let $\{(G_\lambda, c_\lambda) \mid \lambda \in \Lambda\}$ be a family of pointed-groups. Then w is a law of $\prod_{\lambda \in \Lambda} (G_\lambda, c_\lambda)$ if and only if w is a law of every pointed-group (G_λ, c_λ) where $\lambda \in \Lambda$.

Proof: i. Suppose that w is a law of (G, c) . Let (H, c) be a sub pointed-group of (G, c) . Then $w(H, c) \subseteq w(G, c)$. But $w(G, c) = \{1\}$. Therefore, $w(H, c) = \{1\}$. Thus w is a law of (H, c) also.

Now let N be a normal sub group of G . Then by Corollary 2.1.2 we have:

$$w(G/N, cN) = w(G, c)N/N = \{N\}$$

Therefore, w is a law of $(G/N, cN)$ as required.

ii. Suppose that $w = w(y, x_1, x_2, \dots, x_n)$ is a law of every pointed-group (G_λ, c_λ) . Then for all $f_1, f_2, \dots, f_n \in \prod_{\lambda \in \Lambda} G_\lambda$ and for all $\lambda \in \Lambda$ we have:

$$w(c, f_1, \dots, f_n)(\lambda) = w(c(\lambda), f_1(\lambda), \dots, f_n(\lambda)) = 1$$

Thus $(c, f_1, \dots, f_n) = 1$ i.e. w is a law of $\prod_{\lambda \in \Lambda} (G_\lambda, c_\lambda)$

Conversely, suppose that w is a law of $\prod_{\lambda \in \Lambda} (G_\lambda, c_\lambda)$.

Then we have to show that w is a law of (G_λ, c_λ) for all $\lambda \in \Lambda$.

Let $(G, c) = \prod_{\lambda \in \Lambda} (G_\lambda, c_\lambda)$. Then for each $\lambda \in \Lambda$ there is an epimorphism from (G, c) to (G_λ, c_λ) i.e. the projection π_λ of (G, c) onto (G_λ, c_λ) . Therefore, by Theorem 1.2.4 (G_λ, c_λ) is isomorphic to a factor pointed-group of (G, c) . Therefore, by the 1st part of the above lemma, every law of (G, c) is a law of (G_λ, c_λ) for all $\lambda \in \Lambda$.

4.1.7 LEMMA: Let W be a set of words and let (G, c) be a pointed-group. Let N be a normal sub group of G . Then every element w of W is a law of $(G/N, c_N)$ if and only if $W(G, c) \subseteq N$.

Proof: Every element w of W is a law of $(G/N, c_N)$ if and only if $W(G/N, c_N) = \{N\}$. Now by Corollary 2.1.2 we have $W(G/N, c_N) = W(G, c)N/N$. Therefore, $W(G/N, c_N) = \{N\}$ if and only if $W(G, c) \subseteq N$.

§ 2.2 VARIETIES OF POINTED-GROUPS

A variety of pointed-groups is the class of all pointed-groups satisfying each one of a given set W of laws. The variety corresponding to a set of words W is called the variety defined by W and is denoted by $\text{var } W$ or \underline{W} . Thus we have:

2.2.1 LEMMA: A pointed-group (G,c) belongs to the variety defined by a set of words W if and only if $W(G,c) = \{1\}$.

An immediate consequence of Lemma 2.1.7 is

2.2.2 LEMMA: Let \underline{W} be a variety defined by a set of words W . Let (G,c) be a pointed-group. Let N be a normal sub group of G . Then $(G/N, cN) \in \underline{W}$ if and only if $W(G,c) \subseteq N$.

Let H be the normal closure of $W(G,c)$ in G . Then $(G/H, cH)$ is the largest factor pointed-group of (G,c) which lies in the variety defined by the set of words W .

EXAMPLES: Let $G = \langle a, b \mid a^3 = b^2 = 1, b^{-1}ab = a^{-1} \rangle$. Take $W = \{y^3\}$ and consider (G,b) . Now the only value of y^3 in (G,b) is the element $b^3 = b$. Thus $W(G,b) = \langle b \rangle$. But the normal closure of $\langle b \rangle$ in G is G itself. Thus the trivial factor pointed-group $(G/G, bG)$ is the largest factor pointed-group of (G,b) which lies in the variety defined by the set of words W .

2.2.3 THEOREM: Let \underline{W} be a variety defined by a set of words W . A pointed-group (G,c) belongs to \underline{W} if and only if (G,c) is isomorphic to a factor pointed-group of some free pointed-group (F,t) with respect to a normal sub group of F containing $W(F,t)$.

Proof: Suppose that $(G,c) \in \underline{W}$. Then by Theorem 1.4.3 (G,c) is isomorphic to a factor pointed-group $(F/R, tR)$ of some free

pointed-group (F,t) of rank greater than or equal to cardinality of a generating set of (G,c) . Now since $(G,c) \in \underline{W}$, by Lemma 2.2.2, we have $W(F,t) \subseteq R$.

Conversely, suppose that (G,c) is isomorphic to a factor pointed-group $(F/R,tR)$ of some free pointed-group (F,t) where $W(F,t) \subseteq R$. Then by Lemma 2.2.2 $(F/R,tR) \in \underline{W}$. Thus $(G,c) \in \underline{W}$.

We shall now examine the conditions under which two sets of words V and W determine the same variety.

Suppose V is a set of words and w is a word. We shall say that the word w is a consequence of V if w is a law of every pointed-group in the variety defined by V i.e. $\text{var } V$.

Now two sets of words V and W are said to be equivalent if every element of V is a consequence of W and if every element of W is a consequence of V . Thus clearly V and W are equivalent if and only if $\text{var } V = \text{var } W$.

EXAMPLE: $\{y^3, [x_1^2, x_2^2]\}$ is a consequence of $\{y^3, [x_1^2, x_2^2]\}$

Proof: Let $(G,c) \in \text{var } \{y^3, [x_1^2, x_2^2]\}$. Then y^3 and $[x_1^2, x_2^2]$ are laws of (G,c) . Now to show that $[y^3, x_1^2]$ is a consequence of $\{y^3, [x_1^2, x_2^2]\}$, it is enough to show that $[y^3, x_1^2]$ is a law of (G,c) .

Now, since y^3 is a law of (G,c) , so $c^3 = 1$ i.e. $c=c^4$. Also $[x_1^2, x_2^2]$ is a law of (G,c) therefore, for any element g_1 of G , we have:

$$[g_1^2, (c^2)] = 1$$

i.e.

$$[g_1^2, c^4] = 1$$

$$\text{i.e.} \quad [g_1^2, c] = 1$$

$$\text{i.e.} \quad [c, g_1^2] = 1$$

Thus $[y, x_1^2]$ is a law of (G, c) as required.

A simple description of equivalence is that two words obtained from each other by renumbering the variables are equivalent.

2.2.4 LEMMA: Let V be a set of words and let w be a word. Then w is a consequence of V if and only if $w \in \hat{V}$.

Proof: First suppose that w is a consequence of V . Then w is a law of every pointed-group (G, c) in $\text{var } V$.

As we have seen that $\bar{V} = V(X, y)$. Therefore, $V(X, y) \subseteq \hat{V}$. Thus by Lemma 2.2.2 we have, $(X/\hat{V}, y/\hat{V}) \in \text{var } V$. Therefore, $w(X/\hat{V}, y/\hat{V}) = (\hat{V})$ i.e. w is a law of $(X/\hat{V}, y/\hat{V})$. Thus again by Lemma 2.2.2, $w(X, y) \subseteq \hat{V}$. Therefore, in particular, $w \in \hat{V}$.

Conversely, suppose that $w \in \hat{V}$. Then by corollary 1.5.4 we can write w in the form

$$w = ((v_1 \alpha_1)^{w_1})^{\epsilon_1} \dots ((v_n \alpha_n)^{w_n})^{\epsilon_n}$$

where $v_i \in V$, $\alpha_i \in \text{End}(X, y)$, $w_i \in X$ and $\epsilon_i = \pm 1$ ($i=1, 2, \dots, n$).

Now let (G, c) be any pointed-group in $\text{var } V$ and let β be a homomorphism from (X, y) to (G, c) . We want to show that

w is a law of (G, c) . To do this, it is enough to show that $w^\beta = 1$. But $w^\beta = ((v_1 \alpha_1^\beta)^{w_1^\beta})^{\epsilon_1} \dots ((v_n \alpha_n^\beta)^{w_n^\beta})^{\epsilon_n}$. Therefore, $w^\beta = 1$, because v_1, v_2, \dots, v_n are the laws of (G, c) and α_i^β ($i=1, \dots, n$) are homomorphisms from (X, y) to (G, c) . Thus the result follows.

2.2.5 COROLLARY: Two sets of words V and W are equivalent i.e. $\text{var } V = \text{var } W$ if and only if $\hat{V} = \hat{W}$.

2.2.6 THEOREM: There is one-one correspondance between the set of normal admissible sub groups of (X, y) and the set of varieties of pointed-groups given by $\hat{V} \leftrightarrow \text{var } \hat{V}$ for each normal admissible sub group \hat{V} of (X, y) .

Proof: Let $\text{var } W$ be any variety of pointed-groups. Then by corollary 2.2.5, we have $\text{var } W = \text{var } \hat{W}$. Therefore, every variety is defined by some normal admissible sub-group of (X, y) .

Suppose \hat{V} and \hat{W} are normal admissible sub groups of (X, y) such that $\text{var } \hat{V} = \text{var } \hat{W}$. Then by corollary 2.2.5 $\hat{V} = \hat{W}$ i.e. $\hat{V} = \hat{W}$. Thus different normal admissible sub groups of (X, y) define different varieties.

REMARKS: The normal admissible sub group of (X, y) corresponding to $\text{var } V$ is \hat{V} .

Suppose \underline{V} is a variety of pointed-groups, then the

corresponding normal admissible sub group of (X,y) is characterized by \underline{V} as follows:

2.2.7 LEMMA: Let \underline{V} be a variety of pointed-groups. Then the normal admissible sub group \acute{V} of (X,y) which defines \underline{V} is the set of all words which are laws of all pointed-groups of \underline{V} .

Proof: Let W be the set of all words which are laws of all pointed groups of \underline{V} . Now since $\underline{V} = \text{var } \acute{V}$, so W is the set of all consequences of \acute{V} . Therefore, by Lemma 2.2.4 we have $W = \hat{\acute{V}} = \acute{V}$ as required.

Now suppose \acute{V} and \acute{W} are normal admissible sub groups of (X,y) such that $\acute{V} \subseteq \acute{W}$. Then clearly $\text{var } \acute{W} \subseteq \text{var } \acute{V}$.

Conversely suppose that $\text{var } \acute{W} \subseteq \text{var } \acute{V}$, then Lemma 2.2.7 shows that $\acute{V} \subseteq \acute{W}$. Thus the one-one correspondance of the Theorem 2.2.6 is "inclusion reversing".

A variety \underline{W} which is contained in another variety \underline{V} is called a *sub variety* of \underline{V} .

Let \underline{V} be any variety of pointed-groups. Let V be the corresponding normal admissible sub group of (X,y) which defines \underline{V} . Then we may define $\underline{V}(G,c)$ to be the word sub group $V(G,c)$ for any pointed-group (G,c) .

2.2.8 LEMMA: Let (G,c) be a pointed-group and let $\underline{V} = \text{var } W$.

Then $\underline{V}(G, c) = W(G, c)^G$.

Proof: Clearly $W(G, c) \subseteq W(G, c)^G$. Now let $W(G, c)^G = N$. Then by Lemma 2.2.2 we have, $(G/N, cN) \in \text{var } W$. But $\underline{V} = \text{var } W = \text{var } V$ where V is the normal admissible sub group of (X, y) corresponding to \underline{V} . Therefore, again by Lemma 2.2.2 we have $V(G, c) \subseteq N$ i.e. $\underline{V}(G, c) \subseteq N$. Now it only remains to show that $N \subseteq V(G, c)$. To do this, we shall prove that $V(G, c)$ is normal in G .

Now by definition $V(G, c)$ is generated by values in (G, c) of the elements of V . Therefore, every element say g of $V(G, c)$ can be written in the form

$$g = (v_1 \alpha_1)^{\epsilon_1} \dots (v_n \alpha_n)^{\epsilon_n}$$

where $v_i \in V$ and α_i are homomorphisms from (X, y) to (G, c) for $i = 1, 2, \dots, n$.

Now let t be any element of G , then to prove that $V(G, c)$ is normal in G , it is enough to show that $t^{-1}gt \in V(G, c)$. But, $t^{-1}gt = (t^{-1}(v_1 \alpha_1)^{\epsilon_1} t) \dots (t^{-1}(v_n \alpha_n)^{\epsilon_n} t)$. Thus, it is enough to show that $(v\alpha)^t \in V(G, c)$ for all $v \in V$, $t \in G$ and α any homomorphism from (X, y) to (G, c) .

Suppose $v = v(y, x_1, x_2, \dots, x_k)$ and suppose $v\alpha = v(c, g_1, g_2, \dots, g_k)$ where $g_1, g_2, \dots, g_k \in G$. Then $(v\alpha)^t = v(c, g_1, g_2, \dots, g_k)^t$ and this is a value of $v(y, x_1, x_2, \dots, x_k)^{x_{k+1}}$ where $x_{k+1}\alpha = t$ for some $x_{k+1} \in X$. But V is normal in X . Therefore, $v(y, x_1, x_2, \dots, x_k)^{x_{k+1}}$ belongs to V . Therefore, $(v\alpha)^t \in V(G, c)$. Therefore, $V(G, c)$ is normal in G . Thus $W(G, c)^G \subseteq V(G, c)$, because $W(G, c) \subseteq V(G, c)$ and

$V(G,c)$ is normal in G . Hence $\underline{V}(G,c) = W(G,c)^G$ as required.

The following are the consequences of Lemma 2.2.8.

2.2.9 COROLLARY: Let \underline{V} be any variety and let V be the corresponding normal admissible sub group of (X,y) . Then $\underline{V}(G,c)$ is normal in G for any pointed-group (G,c) .

2.2.10 COROLLARY: Let \underline{V} be a variety of pointed-groups corresponding to a normal admissible sub group V of (X,y) and let (G,c) be a pointed-group. Then $(G/\underline{V}(G,c), c \underline{V}(G,c))$ is the largest factor pointed-group of (G,c) which lies in the variety \underline{V} .

Let (F,t) be an arbitrary free pointed-group with a generating set S and let \underline{V} be a variety. Then by corollary 2.2.10 $(F/\underline{V}(F,t), t\underline{V}(F,t))$ is the largest factor pointed-group of (F,t) which lies in the variety \underline{V} . The pointed-group $(F/\underline{V}(F,t), t\underline{V}(F,t))$ is called the *relatively free pointed-group* of \underline{V} on S and is denoted by $(F,t)(\underline{V})$.

The cardinality of S is called rank of $(F,t)(\underline{V})$. Clearly two sets S_1 and S_2 of the same cardinality give isomorphic relatively free pointed-groups in \underline{V} .

Relatively free pointed-groups have properties

analogous to the properties of free pointed-groups described in the previous section §1.6. Of these we shall state and prove the analogues of Theorem 1.6.4 and corollary 1.6.6.

2.2.11 THEOREM: Let (G, c) be any pointed-group in a variety \underline{V} with a generating set of cardinality less than or equal to the rank of a free pointed-group (F, t) . Then (G, c) is isomorphic to a factor pointed-group of $(F, t)(\underline{V})$.

Proof: By Theorem 1.6.4 there is a normal sub group R of F such that $(G, c) \cong (F/R, tR)$.

Now since $(G, c) \in \underline{V}$, so $(F/R, tR) \in \underline{V}$. Therefore, by Lemma 2.2.2 we have $\underline{V}(F, t) \subseteq R$. But $\underline{V}(F, t)$ is also normal in F (by corollary 2.2.9). Thus by Theorem 1.2.5 we have:

$$(F/R, tR) \cong (F_1/N, t_1N)$$

where $F_1 = F/\underline{V}(F, t)$, $N = R/\underline{V}(F, t)$ and $t_1 = t\underline{V}(F, t)$. Therefore, $(G, c) \cong (F_1/N, t_1N)$. But $(F_1/N, t_1N)$ is a factor pointed-group of $(F, t)(\underline{V})$. Thus the theorem follows.

2.2.12 THEOREM: Let (F, t) be a free pointed-group freely generated by a set S . Let \underline{V} be a variety of pointed-groups and write $N = \underline{V}(F, t)$. Let $\acute{S} = \{sN \mid s \in S\}$. Then any mapping $\alpha: \acute{S} \rightarrow F/N$ can be extended to an endomorphism α^* of the relatively free pointed-group $(F/N, tN)$.

Proof: Let $\beta: S \rightarrow F$ be chosen so that $(s\beta)N = (sN)$ for all

$s \in S$. By corollary 1.6.6, there is an endomorphism β^* of (F, t) which extend β . Since N is a normal admissible sub group of (F, t) we have $N\beta^* \subseteq N$. Thus we can define an endomorphism α^* of $(F/N, tN)$ by $(fN)\alpha^* = (f\beta^*)N$ for all $f \in F$. It is straight forward to check that α^* extends α . Thus the theorem follows:

Chapters III and IV of this thesis will be devoted to the question of which varieties of pointed groups have a finite basis for their laws. We now explain what this means. As we saw in the previous section that two sets of words V and W define the same variety if and only if $\hat{V} = \hat{W}$ i.e. V and W have the same closure in (X, y) .

Now any set of words which defines a variety \underline{V} is called a *basis* for (the laws of) \underline{V} . A variety which has a finite basis is called *finitely based*. Thus if V denotes the set of all laws of \underline{V} then \underline{V} is finitely based variety if and only if V is the closure of some finite subset of words.

§ 2.3 BIRKHOFF'S THEOREM

Let \underline{Z} be any class of pointed-groups. Then $Q\underline{Z}$ denotes the class of all pointed-groups isomorphic to a factor pointed-group of a pointed-group in the class \underline{Z} ; $S\underline{Z}$ the class of all pointed-group isomorphic to a sub pointed-group of a pointed-group in \underline{Z} and $C\underline{Z}$ denotes the class of all pointed-groups isomorphic to a cartesian product of pointed-groups in \underline{Z} .

We say that \underline{Z} is a (Q,S,C) closed class, if $Q\underline{Z} \subseteq \underline{Z}$, $S\underline{Z} \subseteq \underline{Z}$ and $C\underline{Z} \subseteq \underline{Z}$ - that is if \underline{Z} is closed under the formation of factor pointed-groups, sub pointed-groups and cartesian products and pointed-groups isomorphic to these.

An immediate consequence of the Lemma 2.1.6 is that every variety of pointed-groups is (Q,S,C) closed. The main result of this section which we want to prove is that every non-empty (Q,S,C) - closed class of pointed-groups is a variety. This fact was first established by Garrett Birkhoff in [1], for rather more general systems. We shall split the proof into several steps:

2.3.1 LEMMA: Let \underline{Z} be a class of pointed-groups. Let V be the set of all elements of (X,y) which are laws in every pointed-group of the class \underline{Z} . Then V is a normal admissible sub group of (X,y) .

Proof: By Lemma 2.2.4, \hat{V} is the set of all consequences of V . Therefore, every element of \hat{V} is a law of every pointed-group in \underline{Z} . Thus, by the choice of V we have $V = \hat{V}$. Hence V is a normal admissible sub group of (X,y) as required.

2.3.2 THEOREM: Let (A,a) be a pointed-group. Let V be the set of all laws of (A,a) . Let \underline{V} be the variety defined by V . Then the relatively free pointed-group $(F,t)(\underline{V})$ of \underline{V} on any set S

isomorphic to a sub pointed-group of the cartesian power $(A, a)^\Lambda$ where Λ is the set of all functions from S to A .

Proof: We shall write c for the focus of (A, a) . Therefore, $c(\lambda) = a$ for all $\lambda \in \Lambda$.

We shall define, for each $s \in S$, a certain element f_s of A^Λ . To do this we must specify an element $f_s(\lambda)$ of A for all $\lambda \in \Lambda$. But, by the definition of Λ , λ is a function from S to A . Therefore, we may define $f_s(\lambda) = \lambda(s)$ for all $\lambda \in \Lambda$. Now by Theorem 1.6.2 we can define a homomorphism $\alpha : (F, t) \rightarrow (A, a)$ by taking $s\alpha = f_s$ for all $s \in S$. Now to prove the theorem we shall prove that the kernel of α is $\underline{V}(F, t)$. It will then follow by Theorem 1.4.3 that $(F/\underline{V}(F, t), t\underline{V}(F, t))$ is isomorphic to the image of α . Thus we shall have proved that $(F, t)/\underline{V}$ is isomorphic to a sub pointed-group of $(A, a)^\Lambda$.

Now let $\ker \alpha = N$. Then $(F/N, tN)$ is isomorphic to a sub pointed-group of $(A, a)^\Lambda$. But by the definition of \underline{V} we have $(A, a) \in \underline{V}$. Therefore, by Lemma 2.1.6, $(F/N, tN) \in \underline{V}$. Thus by Lemma 2.2.2, $\underline{V}(F, t) \subseteq N$. Thus to finish the proof it only remains to prove that $N \subseteq \underline{V}(F, t)$.

Now every element of (F, t) can be written in the form

$$w(t, s_1, s_2, \dots, s_n)$$

where s_1, s_2, \dots, s_n are distinct elements of S .

Suppose that $w(t, s_1, s_2, \dots, s_n) \in N$ and let $w = w(y, x_1, x_2, \dots, x_n)$ be the corresponding element of (X, y) . To complete the proof we shall show that $w(t, s_1, s_2, \dots, s_n)$ belongs to $\underline{V}(F, t)$. To do this, we shall show that w is a law of (A, a) where $w \in V$.

Now let a_1, a_2, \dots, a_n be the elements of A . Let λ be a function from S to A defined by $\lambda(s_i) = a_i$ ($i=1, 2, \dots, n$) and $\lambda(s) = 1$ for all other $s \in S$. Then consider

$$\begin{aligned} w(a, a_1, \dots, a_n) &= w(a, \lambda(s_1), \dots, \lambda(s_n)) \\ &= w(c(\lambda), f_{s_1}(\lambda), \dots, f_{s_n}(\lambda)) \\ &= w(c, f_{s_1}, \dots, f_{s_n})(\lambda) \end{aligned}$$

$$\begin{aligned} \text{But, } w(c, f_{s_1}, f_{s_2}, \dots, f_{s_n}) &= w(t\alpha, s_1\alpha, \dots, s_n\alpha) \\ &= w(t, s_1, s_2, \dots, s_n)\alpha \\ &= 1 \end{aligned}$$

Thus $w(c, f_{s_1}, \dots, f_{s_n})(\lambda) = 1$. Hence, $w(a, a_1, \dots, a_n) = 1$. Therefore, w is a law of (A, a) as desired.

2.3.3 THEOREM: Let \underline{Z} be any non-empty class of pointed-groups. Then $\text{QSC}\underline{Z}$ is a variety.

Proof: Let V be the set of all laws of all pointed-groups in \underline{Z} . Let $\underline{V} = \text{var } V$. We want to prove that $V = \text{QSC}\underline{Z}$.

Now clearly by Lemma 2.1.6 we have $\text{QSC}\underline{Z} \subseteq \underline{V}$. Thus to complete the proof, it only remains to prove that $\underline{V} \subseteq \text{QSC}\underline{Z}$. For each element w of X/V ; let (G_w, c_w) be a pointed-group in \underline{Z} which does not have w as a law. (Notice that such a

pointed-group exists by the definition of V). Let (A, a) be the cartesian product of the pointed-groups (G_w, c_w) . Thus by Lemma 2.1.6, every element of V is a law of (A, a) . But again by Lemma 2.1.6, no element of X/V is a law of (A, a) , because w is not a law of (G_w, c_w) . Thus V is precisely the set of all laws of (A, a) .

Now let (G, c) be any pointed-group in \underline{V} . Then by Theorem 2.2.11, (G, c) is isomorphic to a factor pointed-group of some relatively free pointed-group of \underline{V} which by Theorem 2.3.2 is isomorphic to a sub pointed-group of some cartesian power $(A, a)^\Lambda$ of (A, a) .

Now since (A, a) is a cartesian product of pointed-groups in \underline{Z} , so is $(A, a)^\Lambda$. Therefore, $(A, a)^\Lambda \in \underline{CZ}$. Thus $(G, c) \in \underline{QSCZ}$, Hence $\underline{V} \subseteq \underline{QSCZ}$ as desired and the result follows.

We are now ready to prove the Theorem of Garrett Birkhoff for pointed-groups. In fact it is an easy consequence of the Theorem 2.3.3.

2.3.4 THEOREM (BIRKHOFF): A non-empty class of pointed-groups is a variety if and only if it is (Q, S, C) - closed.

Proof: As we have already observed, Lemma 2.1.6 shows that every variety is (Q, S, C) -closed.

Conversely, suppose that \underline{Z} is a non-empty (Q, S, C) - closed class of pointed-groups. Then $\underline{QSCZ} \subseteq \underline{Z}$. But $\underline{Z} \subseteq \underline{QSCZ}$

holds for every class \underline{Z} of pointed-groups. Thus we have $\underline{Z} = \text{QSC}\underline{Z}$. Hence by Theorem 2.3.3, \underline{Z} is a variety.

Now for any non-empty class \underline{Z} of pointed-groups, $\text{QSC}\underline{Z}$ contains \underline{Z} and is a variety by Theorem 2.3.3. Also, every variety containing \underline{Z} must contain $\text{QSC}\underline{Z}$; because varieties are (Q,S,C) -closed. Thus $\text{QSC}\underline{Z}$ is called the *variety generated by \underline{Z}* and is denoted by $\text{var } \underline{Z}$. Now suppose \underline{Z} is the empty class. Then we define $\text{var } \underline{Z}$ to be the variety of all *trivial pointed-groups*. If (G,c) is a pointed-group, then variety generated by $\{(G,c)\}$ is called the *variety generated by (G,c)* and denoted by $\text{var}(G,c)$.

A word which is a law of every member of a class of pointed-groups \underline{Z} is called a *law of \underline{Z}* . Thus Lemma 2.2.7 stated that the normal admissible sub group of (X,y) corresponding to a variety \underline{V} may be characterized as the set of all laws of \underline{V} . Thus we have:

2.3.5 THEOREM: Let \underline{V} be the variety generated by a class of pointed-groups \underline{Z} . Then the normal admissible sub group of (X,y) corresponding to \underline{V} is the set of all laws of \underline{Z} .

Proof: Let V be the normal admissible sub group of (X,y) corresponding to \underline{V} . Then, by Lemma 2.2.7, V is the set of all laws of \underline{V} .

Let W be the set of all laws of \underline{Z} . Since $\underline{V} = \text{var } \underline{Z}$ we have, $\underline{Z} \subseteq \underline{V}$. Therefore, $V \subseteq W$. Now if \underline{Z} is empty, then the result is trivial so we assume that \underline{Z} is non-empty. Then $\underline{V} = \text{QSCZ}$ and Lemma 2.1.6, yields that $W \subseteq V$. Thus $V=W$ as required.

2.3.6 THEOREM: Every variety is generated by the finitely generated pointed-groups it contains.

Proof: Let \underline{Z} be the class of all finitely generated pointed-groups in a variety \underline{V} . Then clearly every law of \underline{V} is a law of \underline{Z} . Thus to prove that $\underline{V} = \text{var } \underline{Z}$ it is enough (by Theorem 2.3.5) to prove that every law of \underline{Z} is a law of \underline{V} .

Now let $w = w(y, x_1, \dots, x_n)$ be a law of \underline{Z} , let $(G, c) \in \underline{V}$ and let $w(c, g_1, \dots, g_n)$ where $g_1, g_2, \dots, g_n \in G$, be the value of w in (G, c) . Let (H, c) be the sub pointed-group of (G, c) generated by $\{g_1, g_2, \dots, g_n\}$. Then (H, c) is a pointed-group in \underline{Z} . Therefore, w is a law of (H, c) . It follows that $w(c, g_1, \dots, g_n) = 1$. Thus w is a law of (G, c) . Hence the theorem follows.

2.3.7 COROLLARY: Every variety \underline{V} is generated by its relatively free pointed-group $(X/V, yV)$ of countable rank where $V = \underline{V}(X, y)$ is the normal admissible sub group of (X, y) which defines \underline{V} . In other words $\underline{V} = \text{var}((X, y)(\underline{V}))$.

Proof: Let \underline{W} be the variety generated by $(X/V, yV)$. Then $(X/V, yV) \in \underline{W}$. Thus by Lemma 2.2.2, we have $\underline{W}(X, y) \subseteq V$.

Now since $\underline{W}(X, y)$ is the set of all laws of \underline{W} , and V is the set of all laws of \underline{V} , it follows that $\underline{V} \subseteq \underline{W}$. But, $(X/V, yV) \in \underline{V}$, so we have $\underline{W} \subseteq \underline{V}$. Thus $\underline{V} = \underline{W}$ as required.

An immediate consequence of this result is that every variety is generated by a single pointed-group.

§ 2.4 INTERSECTION AND JOIN VARIETIES

Suppose \underline{U} and \underline{V} are pointed-groups varieties, then clearly $\underline{U} \cap \underline{V}$ is (Q, S, C) -closed and so is a variety. We write $\underline{U} \wedge \underline{V}$ to denote the intersection of \underline{U} and \underline{V} . Thus we have:

2.4.1 THEOREM: If U is the normal admissible sub group of (X, y) corresponding to the variety \underline{U} and V is the normal admissible sub group of (X, y) corresponding to the variety \underline{V} . Then the normal admissible sub group of (X, y) corresponding to $\underline{U} \wedge \underline{V}$ is UV .

Proof: Clearly UV is a normal admissible sub group of (X, y) . Thus, to prove the theorem, it is sufficient to prove that $\underline{U} \wedge \underline{V} = \text{var } UV$.

Now $U \subseteq UV$ and $V \subseteq UV$. Thus $\text{var } UV \subseteq \underline{U}$ and $\text{var } UV \subseteq \underline{V}$. Hence $\text{var } UV \subseteq \underline{U} \wedge \underline{V}$.

On the other hand, if $(G, c) \in \underline{U} \wedge \underline{V}$, then every law

of (G, c) is an element of UV . It follows that $\underline{U} \wedge \underline{V} \subseteq \text{var } UV$. Hence, $\underline{U} \wedge \underline{V} = \text{var } UV$, as required.

Now suppose that \underline{U} and \underline{V} are pointed-groups varieties. Then we write $\underline{U} \vee \underline{V}$ to denote the variety generated by $\underline{U} \cup \underline{V}$, that is, $\text{var}(\underline{U} \cup \underline{V})$. Thus we have:

2.4.2. THEOREM: Suppose U is the normal admissible sub group of (X, y) corresponding to the variety \underline{U} and V is the normal admissible sub group of (X, y) corresponding to the variety \underline{V} . Then the normal admissible sub group of (X, y) corresponding to $\underline{U} \vee \underline{V}$ is $U \cap V$.

Proof: Let W be the normal admissible sub group of (X, y) corresponding to the variety $\underline{U} \vee \underline{V}$. We want to prove that $W = U \cap V$.

Now since $\underline{U} \vee \underline{V}$ contains \underline{U} and $\underline{U} \vee \underline{V}$ contains \underline{V} , it follows that $W \subseteq U$ and $W \subseteq V$, which yields that $W \subseteq U \cap V$. Thus to complete the proof it only remains to prove that $U \cap V \subseteq W$. To do this, let $w \in U \cap V$. Then $w \in U$ and $w \in V$. Therefore, w is a law of \underline{U} and also w is a law of \underline{V} . Thus, it follows that w is a law of $\underline{U} \cup \underline{V}$. But, $\text{var}(\underline{U} \cup \underline{V}) = \underline{U} \vee \underline{V}$. Thus by Theorem 2.3.5, w is a law of $\underline{U} \vee \underline{V}$. Hence $w \in W$. Thus $U \cap V \subseteq W$, as desired and the result thus follows.

§ 2.5 THE n-VARIABLE LAWS OF A POINTED-GROUP VARIETY

For each positive integer n , let $(X_n, Y) = \langle\langle x_1, \dots, x_n \rangle\rangle$ denote the sub pointed-group of (X, Y) generated by the set $\{x_1, x_2, \dots, x_n\}$. Clearly (X_n, Y) is a free pointed-group on the set $\{x_1, x_2, \dots, x_n\}$, because X_n is a free group on $\{y, x_1, x_2, \dots, x_n\}$.

A word w which belongs to (X_n, Y) is called an *n-variable word*. Notice that an *n-variable word* is specifically a word in the variables y and x_1, x_2, \dots, x_n .

Let \underline{V} be any pointed-group variety and let $V = \underline{V}(X, Y)$ be the set of all laws of \underline{V} , that is, the normal admissible sub group of (X, Y) corresponding to \underline{V} . Then the set denoted by $V_n = V \cap X_n$ is the set of all laws of \underline{V} which belongs to (X_n, Y) -that is the set of all *n-variable laws* of \underline{V} .

Moreover, for each positive integer n , let $\underline{V}^{(n)}$ denote the variety defined by V_n , that is, $\underline{V}^{(n)}$ is the variety defined by the set of all *n-variable laws* of \underline{V} .

Now since, $V \cap X_1 \subseteq V \cap X_2 \subseteq \dots \subseteq V \cap X_n \subseteq \dots \subseteq V$
and
$$V = \bigcup_{n=1}^{\infty} (V \cap X_n),$$

it follows that $\underline{V}^{(1)} \supseteq \underline{V}^{(2)} \supseteq \dots \supseteq \underline{V}^{(n)} \supseteq \dots \supseteq \underline{V}$ and $\underline{V} = \bigcap_{n=1}^{\infty} \underline{V}^{(n)}$

2.5.1 LEMMA: Let \underline{V} be a variety of pointed-groups. Then, for each positive integer n , $\underline{V}^{(n)}(X_n, Y) = \underline{V}(X_n, Y) = \underline{V}(X, Y) \cap X_n$.

Proof: Since $\underline{V} \subseteq \underline{V}^{(n)}$, we have $\underline{V}^{(n)}(X_n, Y) \subseteq \underline{V}(X_n, Y)$. Also since $\underline{V}(X_n, Y) \subseteq \underline{V}(X, Y)$, so we have $\underline{V}(X_n, Y) \subseteq \underline{V}(X, Y) \cap X_n$. Thus to prove the Lemma, it only remains to prove that $\underline{V}(X, Y) \cap X_n \subseteq \underline{V}^{(n)}(X_n, Y)$. To do this, let $w \in \underline{V}(X, Y) \cap X_n$. Then w is an n -variable law of \underline{V} . Hence w is a law of $\underline{V}^{(n)}$. Clearly w is a value of w in (X_n, Y) . Thus $w \in \underline{V}^{(n)}(X_n, Y)$. Thus the Lemma follows.

In the previous section, corollary 1.6.5 shows that every n -generator pointed-group is isomorphic to a factor pointed-group of (X_n, Y) . Thus, by the proof of Theorem 2.2.11, an n -generator pointed-group in a variety \underline{V} is isomorphic to a factor pointed-group of $(X_n / \underline{V}(X_n, Y), Y \underline{V}(X_n, Y))$.

2.5.2 LEMMA: An n -generator pointed-group belongs to the variety \underline{V} if and only if it satisfies the n -variable laws of \underline{V} .

Proof: Only if is trivial. For the converse, suppose (G, c) is an n -generator pointed-group belonging to $\underline{V}^{(n)}$. Then by the comment above, (G, c) is isomorphic to a factor pointed-group of $(X_n / N, YN)$ where $N = \underline{V}^{(n)}(X_n, Y)$. But, $\underline{V}^{(n)}(X_n, Y) = \underline{V}(X_n, Y)$ [by Lemma 2.5.1]. Thus $(G, c) \in \underline{V}$ as required.

The next theorem characterises $\underline{V}^{(n)}$ as the class of all those pointed-groups (G, c) whose n -generator sub pointed-groups are all in \underline{V} .

2.5.3 **THEOREM:** Let \underline{V} be a variety of pointed-groups. Let n be a positive integer. Then a pointed-group (G,c) belongs to $\underline{V}^{(n)}$ if and only if every n -generator sub pointed-group of (G,c) belongs to \underline{V} .

Proof: Suppose $(G,c) \in \underline{V}^{(n)}$. Let (H,c) be an n -generator sub pointed-group of (G,c) . Then by Lemma 2.5.2, we have $(H,c) \in \underline{V}$.

Conversely, suppose that (G,c) is a pointed-group such that every n -generator sub pointed-group of (G,c) belongs to \underline{V} . We want to prove that $(G,c) \in \underline{V}^{(n)}$. To do this, let w be an n -variable law of \underline{V} . Then every value of w in (G,c) is a value of w in some n -generator sub pointed-group of (G,c) and so is trivial. Therefore, it follows that $(G,c) \in \underline{V}^{(n)}$ as desired.