Chapter 1

STRUCTURE OF POINTED - GROUPS

Chapter I

STRUCTURE OF POINTED-GROUPS

§ 1.1 DEFINITIONS AND NOTATION

An ordered pair (G,c) consisting of a group G and an element c of G is called a *pointed-group*. Call G the *carrier* of (G,c) and c is the *focus* of (G,c).

A pointed-group (G,c) may be thought of as a group G with an extra nullary-operation on G which picks out a selected element c of G.

Let (G,c) and (H,d) be two given pointed-groups. We say that (H,d) is a *sub pointed-group* of (G,c) and write (H,d) ≤ (G,c), if the carrier H of (H,d) is the sub group of the carrier G of (G,c) and the focus d of (H,d) is equal to the focus c of (G,c). Moreover, a sub pointed-group (H,c) of (G,c) is called *proper* and we write (H,c) < (G,c), if H is a proper sub group of G. Further, let (G,c) be any pointed-group. Let N be a normal sub group of G (not necessary containing c). Then the pointed-group (G/N,cN) is called a *factor pointed-group* of (G,c). Here cN denotes the coset of G/N containing c.

Now we come to examine the collection of sub pointed-groups of any pointed-group (G,c).

Suppose (H_1, c) and (H_2, c) are two sub pointed-groups of a pointed-group (G,c). Then obviously $c \in H_1 \cap H_2$ and $H_1 \cap H_2$ is a sub group of G. Therefore by the definition of sub pointed-group, $(H_1 \cap H_2, c)$ is a sub pointed-group of (G,c). We call $(H_1 \cap H_2, c)$ the intersection of the pointed-groups (H_i, c) where i=1,2.

More generally, if $\{(H_i, c) \mid i \in I\}$ where I is an index set, is a family of sub pointed-groups of (G,c). Then we define the intersection of the pointed-groups (H_i, c) as $(\bigcap_{i \in I} H_i, c)$ and this is a sub pointed-group of (G,c). Thus we make the following definition:

<u>**DEFINITION**</u>: Let T be a family of sub pointed-groups of a pointed-group (G,c). Then the intersection $\cap T$ is the sub pointed-group of (G,c) whose carrier is the intersection of the carriers of the members of T.

Now we are ready to define generators of a pointed-group.

Let (G,c) be a pointed-group. Let S be any set of elements of the group G. Let T be the set of all sub pointed-groups of (G,c) whose carriers contain S. Then the intersection, ∩T is a sub pointed-group of (G,c) whose carrier contains S. It is the smallest sub pointed-group of (G,c) whose carrier contains S i.e. it is contained in every other sub pointed-group of (G,c) whose carrier contains S. We call it the sub pointed-group of (G,c) generated by S and denote it by <<S>>.

Thus it follows from this definition that <<S>> = (<S∪{c}>, c) where <S∪{c}> denates the sub group of G generated by S∪{c}.

Now suppose <<S>> = (G,c), then we say that S generates (G,c). Hence from this definition we deduce that:

1.1.1 **LEMMA**: S generates (G,c) if and only if S∪{c} generates G.

Moreover, if (G,c) is generated by a finite set, then we say that (G,c) is finitely generated and if (G,c) is generated by a finite set with n or fewer elements, then we say that (G,c) is an n-generator pointed-group. Thus from Lemma 1.1.1, we have:

1.1.2 **LEMMA**: A pointed-group (G,c) is finitely generated if and only if G is finitely generated.

Now an immediate consequence of Lemma 1.1.2 is:

1.1.3 <u>LEMMA</u>: Suppose (G,c) is a finitely generated pointed-group. Let N be a normal sub group of G. Then (G/N , cN) is finitely generated.

§ 1.2 CARTESIAN PRODUCTS

Let $\{(G_\lambda , c_\lambda) | \lambda \in \Lambda\}$ be a family of pointed-groups where Λ is an index set. We write $\prod_{\lambda\in\Lambda} G_\lambda$ to denote the cartesian product —— of the group G_λ - that is the set of all functions f on Λ which satisfy $f(\lambda) \in G_\lambda$ for all $\lambda\in\Lambda$. Thus we define the cartesian product of the pointed-group (G_λ , c_λ) denoted by $\prod_{\lambda\in\Lambda}(G_\lambda , c_\lambda)$ to be the pointed-groups $(\prod_{\lambda\in\Lambda} G_\lambda , c)$ where c is the function on Λ defined by $c(\lambda) = c_\lambda$ for all $\lambda\in\Lambda$.

One special case of importance to note is that if $(G_\lambda , c_\lambda) = (G,c)$ for all $\lambda\in\Lambda$. Then the cartesian product $\prod_{\lambda\in\Lambda}(G_\lambda , c_\lambda)$ becomes a Cartesian power of (G,c) and is denoted by $(G,c)^\Lambda$.

Now suppose $(G,c) = \prod_{\lambda\in\Lambda} (G_\lambda , c_\lambda)$ is a Cartesian product of pointed-groups (G_λ , c_λ). Let for each $\lambda\in\Lambda$, π_λ denote the function from G to G_λ defined by $f\pi_\lambda = f(\lambda)$ for all $f\in G$.

By the results from group theory π_λ is an epimorphism from G to G_λ - called the projection of G onto G_λ. But $c\pi_\lambda = c(\lambda) = c_\lambda$ for all $\lambda\in\Lambda$. Therefore according to the definition of pointed-groups epimorphism which we shall give below in the next section, π_λ is an epimarphism from (G,c) to (G_λ , c_λ). We shall call it the *projection* of (G,c) onto (G_λ , c_λ).

§ 1.3 POINTED-GROUP HOMOMORPHISM

In order to develop the theory of pointed-groups, the idea of homomorphism is our starting point.

Suppose (A,a) and (B,b) are two pointed-groups. A group homomorphism α from A to B such that $a\alpha=b$, is called a *pointed-group homomorphism* from (A,a) to (B,b).

Suppose α is a homomorphism from (A,a) to (B,b) we write α: (A,a) $\rightarrow$ (B,b). Now we define some important homomorphisms below:

Suppose a homomorphism α from (A,a) to (B,b) is injective then we say that α is a *monomorphism* and that (A,a) is *embeddable* in (B,b).

Suppose a homomorphism α from (A,a) to (B,b) is surjective, then we say that α is an *epimorphism* and that (B,b) is a *homomorphic image* of (A,a).

1.3.1 THEOREM: The product of pointed-groups homomorphism is a homomorphism.

Proof: The proof follows from the result of elementary group theory.

DEFINITION: A homomorphism $\alpha : (G,c) \to (H,d)$ which is one to one and onto is called an *isomorphism* of (G,c) to (H,d).

Suppose there is an isomorphism from (G,c) to (H,d), then we say that (G,c) is *isomorphic* to (H,d) and write $(G,c) \cong (H,d)$.

Note that an isomorphism $\alpha : (G,c) \to (H,d)$ is a group isomorphism $\alpha : G \to H$ such that $c\alpha=d$. Therefore, if $\alpha:(G,c) \to (H,d)$ is an isomorphism of pointed-groups, then the inverse function $\alpha^{-1}: H \to G$ is a group isomorphism (by elementary result of group theory). But since $c\alpha = d$, we have $d\alpha^{-1} = c$. Thus α^{-1} is an isomorphism of (H,d) to (G,c).

§ 1.4 IMAGE AND KERNEL

Suppose $\alpha : (G,c) \to (H,d)$ is a pointed-group homomorphism. Then, in general, α is not surjective. In other words there may be elements of H which are not images of elements of G. We write $G\alpha$ for the set consisting of all images $g\alpha$, $g \in G$. Obviously $c\alpha = d \in G\alpha$ and $G\alpha$ is a sub group of H [by elementary group theory]. Therefore, $(G\alpha, d)$ is a sub pointed-group of (H,d). Thus we make the following definition:

DEFINITION: Let $\alpha : (G,c) \to (H,d)$ be a homomorphism. Then we define the image of α denoted by *Imα* or *(G,c)α* to be the sub pointed-group $(G\alpha,d)$ of (H,d).

Next we see and examine the important fact that every homomorphism of (G,c) is associated with a normal sub group of G.

Suppose $\alpha : (G,c) \to (H,d)$ is a homomorphism. Let K be the set of all those elements of G which are mapped under α to the identity element of H. This set is called the Kernel of α, written as *Ker*α.

Note that Kerα is the same as Kernel of α regarded as a group-homomorphism.

1.4.1 **<u>THEOREM</u>:** Let $\alpha : (G,c) \to (H,d)$ be a homomorphism. Then the kernel of α is a normal sub group of G.

Proof: This is an immediate consequence of the result from elementary group theory.

1.4.2 **<u>THEOREM</u>:** A pointed group homomorphism is injective if and only if Ker$\alpha = \{1\}$.

Proof: The proof follows from the result of elementary group theory.

Let (G,c) be a pointed-group and let N be a normal sub group of G. The function $\alpha : G \to G/N$ defined by $g\alpha = gN$ for all $g \in G$, is a group-epimorphism [by elementary group theory] and we have $c\alpha = cN$. Therefore, $\alpha : (G,c) \to (G/N , cN)$

is a pointed-group epimorphism. It is called the *natural homomorphism* from (G,c) to $(G/N\ ,\ cN)$.

Note that the natural homomorphism α from (G,c) to $(G/N\ ,\ cN)$ is the same as that defined for groups in elementary group theory.

The next result will enable us to determine all the homomorphic images of (G,c) upto isomorphism.

1.4.3 **THEOREM**: Let $\alpha : (G,c) \rightarrow (H,d)$ be a pointed-group homomorphism from (G,c) to (H,d) with $\ker\alpha = K$. Then $(G/K\ ,\ cK) \cong (G\alpha\ ,\ d)$.

Proof: By the Ist isomorphism theorem of elementary group theory, there is a group isomorphism θ from G/K to $G\alpha$ in which $(gK)\theta = g\alpha$ for all $g \in G$. But since $(cK)\theta = c\alpha = d$, so θ is a pointed-group isomorphism, giving the required result.

1.4.4 **THEOREM**: Let (G,c) be a pointed-group and let (A,c) be a sub pointed-group of (G,c). Let N be a normal sub group of G. Then

$$(A/A \cap N\ ,\ c(A \cap N)) \cong (AN/N\ ,\ cN)$$

Proof: By elementary group theory, there is a group-homomorphism θ from A onto AN/N in which $a\theta = aN$ for all $a \in A$.

Clearly θ is a pointed-group homomorphism from (A,c) to $(AN/N, cN)$. Thus, by theorem 1.4.3, we have

$$(A/\mathrm{Ker}\theta\ ,\ c\mathrm{Ker}\theta) = (AN/N\ ,\ cN)$$

But, it is easy to check that Ker $\theta = A \cap N$. Thus the result follows.

1.4.5 __THEOREM__: Let (G,c) be a pointed-group. Let M and N be normal sub groups of G such that N is contained in M. Then $(G_1/M_1\ ,\ c_1M_1) \cong (G/M\ ,\ cM)$ where $G_1 = G/N$ and $M_1 = M/N$ and $c_1 = cN$

Proof: By elementary group theory, there is a group-homomorphism θ from G/N onto G/M in which $(gN)\theta = gM$ for all $g \in G$. Clearly θ is a homomorphism from $(G/N\ ,\ cN)$ to $(G/M\ ,\ cM)$. Therefore, by Theorem 1.4.3, we have $(G_1/\mathrm{Ker}\theta\ ,\ c_1\mathrm{Ker}\theta) \cong (G/M\ ,\ cM)$. But it is easy to see that $\mathrm{Ker}\theta = M/N$. Thus the result follows.

1.4.6 __THEOREM__: Let (G,c) be a pointed-group. Let N be a normal sub group of G. Then there is a one-one correspondence between the sub pointed-groups of $(G/N\ ,\ cN)$ and those sub pointed-groups of (G,c) whose carriers contain N.

Proof: Let θ be the natural homomorphism from (G,c) onto $(G/N\ ,\ cN)$.

Now, by elementary group theory, if S is any sub group of G containing N, then $S\theta$ is a sub group of G/N and if H is any sub group of G/N, then $H\theta^{-1}$ is a sub group of G containing N. In fact there is a bisection Φ from the set of all sub groups of G containing N to the set of all sub groups of G/N in which $S\Phi=S\theta$, for every sub group S of G containing N and $H\Phi^{-1} = H\theta^{-1}$ for every sub group H of G/N. Clearly, if (S,c) is any sub pointed-group of (G,c) then $(S\theta,cN)$ is a sub pointed-group of (G/N , cN).

Now define a function ψ from the set of all sub pointed-groups (S,c) of (G,c) with carrier S containing N to the set of all sub pointed groups of (G/N , cN) by

$$(S,c)\psi = (S\theta,cN)$$

Thus to prove the theorem we shall prove that ψ is a bisection. The fact that ψ is one to one follows from the fact that Φ is one to one. Thus to finish the proof, it only remains to prove that ψ is onto. To do this, suppose that (H,cN) is any sub pointed-group of (G/N , cN). Then H is a sub group of G/N and $c\theta = cN \in H$. Therefore, $c\in H\theta^{-1}$. Also $H\theta^{-1}$ is a sub group of G containing N. Therefore, $(H\theta^{-1}, c)$ is a sub pointed-group of (G,c) whose carrier $H\theta^{-1}$ contains N. Now

$$\begin{aligned}(H\theta^{-1} , c)\psi &= (H\theta^{-1}\theta , cN)\\ &= (H\Phi^{-1}\Phi , cN)\\ &= (H , cN).\end{aligned}$$

Thus the theorem follows.

DEFINITION: Let (G,c) be a pointed-group. By an *endomorphism* of (G,c) we mean a homomorphism α from (G,c) to (G,c) such that $c\alpha = c$.

For example, let $G = \langle a , b | a^3 = b^2 = (ab)^2 = 1\rangle$ Take the element b in G and consider (G,b). Then there is an endomorphism α of G such that $a\alpha = 1$, $b\alpha = b$. But, since $b\alpha = b$, so α is an endomorphism of (G,b).

We write *End(G,c)* to denote the set of all endomorphisms of (G,c).

DEFINITION: Let (G,c) be a pointed-group. By an *automorphism* of (G,c) we mean an isomorphism from (G,c) to (G,c).

For example, let $G = \langle a , b | a^3 = b^2 = (ab)^2 = 1\rangle$. Take the element b in G and consider (G,b). Then there is an automorphism α of G such that $a\alpha = a^2$ and $b\alpha = b$. Now since $b\alpha = b$, so α is an automorphism of (G,b).

§ 1.5 ADMISSIBLE CLOSURE AND CLOSURE IN POINTED GROUPS

We begin by recalling the definition of the normal closure of a subset S of a group G.

Let S be any set of elements of G. Then consider the collection of all normal sub groups of G each of which contains S. This is a non empty collection, because G belongs to it. Write S^G for the intersection of this collection. Therefore, then S^G is a normal sub group of G containing S and

S^G is contained in very other normal sub group of G containing S. We call S^G the *normal closure* of S in G. Thus we have:

1.5.1 **THEOREM**: S^G is the sub group of G generated by all conjugates in G of the elements of S.

Proof: The proof is straight forward and follows from definition.

DEFINITION: Let (G,c) be a pointed-group. Let H be a sub group of G. We say that H is an *admissible* sub group of (G,c) if $H\alpha \leq H$ for every endomorphism α of (G,c).

Now suppose $\{H_\lambda | \lambda \in \Lambda\}$ is a non empty collection of admissible sub groups of a pointed-group (G,c). Let $H=\bigcap_{\lambda\in\Lambda} H_\lambda$. If α is any endomorphism of (G,c) and h is any element of H then we have

$$h\alpha \in \bigcap_{\lambda\in\Lambda} H_\lambda\alpha \leq \bigcap_{\lambda\in\Lambda} H_\lambda$$

because for all $\lambda\in\Lambda$, H_λ is an admissible sub group of (G,c). Therefore, $h\alpha \in H$ and therefore $H\alpha \leq H$. Thus H is an admissible sub group of (G,c). Thus the intersection of any non empty collection of admissible sub groups of (G,c) is again an admissible sub group of (G,c). Thus we have the following definition:

DEFINITION: Let (G,c) be a pointed-group. Let S be any set of elements of G. Then consider the collection of all admissible sub groups of (G,c) each of which contains S. This is a non empty collection, because (G,c) belongs to it. Write $\bar{S}$ for the intersection of this collection. Then $\bar{S}$ is an admissible sub group of (G,c) containing S and $\bar{S}$ is contained in every other admissible sub group of (G,c) containing S. We call $\bar{S}$ the *admissible closure* of S in (G,c). Thus we have:

1.5.2 **THEOREM**: $\bar{S}$ is the sub group of G generated by all the elements of the form $s\alpha$ where $s \in S$ and α is an endomorphism of (G,c).

Proof: Let $S' = \{s\alpha \mid s \in S, \alpha \in End(G,c)\}$. To prove the theorem we have to prove that $\bar{S} = \langle S' \rangle$. To do this, first we shall prove that $\langle S' \rangle$ is an admissible sub group of (G,c). Now every element x of $\langle S' \rangle$ can be written in the form:

$$x = (s_1\alpha_1)^{\epsilon_1}(s_2\alpha_2)^{\epsilon_2} \ldots (s_n\alpha_n)^{\epsilon_n}$$

where $s_1, s_2, \ldots, s_n \in S$ and $\alpha_i \in End(G,c)$, $\epsilon_i = 1$ or -1 $(i=1,2,\ldots, n)$.

Now let β be any endomorphism of (G,c). To prove that $\langle S' \rangle\beta \le \langle S' \rangle$ it is enough to show that $x\beta \in \langle S' \rangle$. But, $x\beta = (s_1\alpha_1\beta)^{\epsilon_1}(s_2\alpha_2\beta)^{\epsilon_2} \ldots (s_n\alpha_n\beta)^{\epsilon_n}$ which is the element of $\langle S' \rangle$, because $s_i \in S$ and $\alpha_i\beta \in End(G,c)$, $\epsilon_i = \pm 1$ $(i=1,2, \ldots, n)$.

Now clearly $S \subseteq \langle S' \rangle$. Therefore, $\overline{S} \subseteq \langle S' \rangle$. But also $\langle S' \rangle \subseteq H$, for each admissible sub group H of (G,c) containing S. Therefore $\langle S' \rangle \subseteq \overline{S}$ and so we have $\overline{S} = \langle S' \rangle$.

Now we know that the intersection of any non empty collection of normal sub groups of G is normal sub groups of G and the intersection of any non empty collection of admissible sub groups of (G,c) is admissible, therefore, it follows that the intersection of any non empty collection of normal admissible sub groups of (G,c) is normal admissible sub group of (G,c). Thus we make the following definition:

DEFINITION: Let (G,c) be a pointed-group. Let S be any set of elements of G. Then consider the collection of all normal admissible sub groups of (G,c) each of which contains S. This is a non empty collection, because (G,c) belongs to it. Write $\hat{S}$ for the intersection of this collection. Therefore, then $\hat{S}$ is a normal admissible sub group of (G,c) containing S and $\hat{S}$ is contained in every other normal admissible sub group of (G,c) containing S. We call $\hat{S}$ the *closure* of S in (G,c). Thus we have.

1.5.3 THEOREM: Prove that $\hat{S} = (\overline{S})^G$.

Proof: To prove the theorem, we have to prove that $(\overline{S})^G \subseteq \hat{S}$ and $\hat{S} \subseteq (\overline{S})^G$.

Now since $\hat{S}$ is an admissible sub group of (G,c) and $\hat{S}$ contains S, so we have $\bar{S} \subseteq \hat{S}$, because $\bar{S}$ is the admissible closure. Also $\hat{S}$ is a normal sub group of G containing S, it follows that $(\bar{S})^G \subseteq \hat{S}$.

To finish the proof it only remains to prove that $\hat{S} \subseteq (\bar{S})^G$. Now since $S \subseteq (\bar{S})^G$, so it is enough to show that $(\bar{S})^G$ is a normal admissible sub group of (G,c). But by definition, $(\bar{S})^G$ is normal. Thus it is enough to show that $(\bar{S})^G$ is an admissible sub group of (G,c). Now by Theorem 1.5.1 every element x of $(\bar{S})^G$ can be written in the form

$$x = ((\bar{s}_1)^{h_1})^{\epsilon_1}((\bar{s}_2)^{h_2})^{\epsilon_2} \ldots. ((\bar{s}_n)^{h_n})^{\epsilon_n}$$

where $\bar{s}_1, \bar{s}_2, \ldots., \bar{s}_n \epsilon \bar{S}$ and $h_i \in G$, $\epsilon_i = \pm 1$ $(i=1,2,\ldots,n)$. Therefore, by Theorem 1.5.2, x takes the form

$$x = ((s_1\alpha_1)^{g_1})^{\epsilon_1} ((s_2\alpha_2)^{g_2})^{\epsilon_2} \ldots. ((s_n\alpha_n)^{g_n})^{\epsilon_n}$$

where $s_1, s_2, \ldots., s_n \in S$ and $\alpha_i \in End(G,c)$, $\epsilon_i = \pm 1$, $h_i\alpha_i = g_i$ $(i= 1,2,\ldots., n)$.

Now let β be any endomorphism of (G,c). Then to prove that $(\bar{S})^G\beta \leq (\bar{S})^G$, it is enough to show that $x\beta \in (\bar{S})^G$. But, $x\beta = ((s_1\alpha_1\beta)^{g_1\beta})^{\epsilon_1} \ldots ((s_n\alpha_n\beta)^{g_n\beta})^{\epsilon_n}$ which is the element of $(\bar{S})^G$, because $\alpha_1\beta, \alpha_2\beta, \ldots., \alpha_n\beta$ are endomorphisms of (G,c). Thus $(\bar{S})^G$ is an admissible sub group of (G,c) as required. Thus the theorem follows:

As a consequence to Theorem 1.5.3 we have:

1.5.4 **<u>COROLLARY</u>:** $\hat{S}$ is the sub group of G generated by all conjugates in G of the elements of the form $s\alpha$, $s \in S$ and $\alpha \in$ End (G,c).

§ 1.6 <u>FREE POINTED-GROUP</u>

A free pointed-group is a pointed-group (F,t) where F is a free group and t is an element of a free generating set of F.

Suppose that F is a free group with a free generating set $S \cup \{t\}$ where $t \notin S$. Then S is called a *free generating set* of (F,t) and the cardinality of S is called the *rank* of (F,t).

In particular, if F is an infinite cyclic group generated by t then (F,t) is a free pointed-group of *rank zero*.

Now since any two free generating sets for a free group F have the same cardinality (equal to the rank of F), so the definition of the rank of (F,t) is independent of a particular choice of S i.e. a free generating set of (F,t). Clearly any two free pointed-groups of the same rank are isomorphic. Thus we have:

1.6.1 **<u>THEOREM</u>:** Two free pointed-groups are isomorphic if any only if they have equal ranks i.e. if and only if their free generating sets have the same cardinality.

1.6.2 <u>**THEOREM**</u>: Let (F,t) be a free pointed-group freely generated by S. Let (G,c) be any pointed-group and let $\alpha : S \to G$ be a mapping. Then there exists a unique homomorphism α^* from (F,t) to (G,c) such that $\alpha^*|S = \alpha$.

Loosely, a homomorphism from (F,t) to (G,c) is determined by assigning arbitrary elements of G to elements of S (and the particular element c of G to t).

Proof: Since (F,t) is freely generated by S, so F is freely generated by $S\cup\{t\}$. Now define a function:

$$\alpha_1 : S\cup\{t\} \to G$$

by $s\alpha_1 = s\alpha$ for all $s \in S$ and $t\alpha_1 = c$. Therefore, α_1 is an extension of the given α such that $\alpha_1|S = \alpha$.

Now by elementary group theory there is a group homomorphism α^* from F to G such that $\alpha^*|S\cup\{t\}=\alpha_1$. But, $t\alpha_1 = c$. Therefore, $t\alpha^* = c$. Therefore, α^* is the required homomorphism from (F,t) to (G,c) such that $\alpha^*|S = \alpha$.

<u>**UNIQUENESS**</u>: Let θ_1 and θ_2 be two homomorphisms from (F,t) to (G,c) such that $\theta_1|S = \theta_2|S$. Then we have $t\,\theta_1 = t\,\theta_2 = c$. But, $S\cup\{t\}$ generates F. Therefore, $a\theta_1 = a\theta_2$ for all $a \in F$. Therefore, $\theta_1 = \theta_2$ as required.

1.6.3 <u>**THEOREM**</u>: Let (F,t) be a free pointed-group. Let β be a homomorphism from (F,t) to (B,b) and let α be an epimorphism from (A,a) to (B,b). Then there exists a homomorphism θ from

(F,t) to (A,a) such that $\theta\alpha=\beta$

Proof:

$$\begin{array}{ccc} (F,t) & \xrightarrow{\beta} & (B,b) \\ & \theta\searrow & \uparrow \alpha \\ & & (A,a) \end{array}$$

Suppose S is a free generating set of (F,t). Now for each $s\in S$ we have $s\beta\in B$ and since α is an epimorphism, so there is an element a_s of A such that $a_s\alpha=s\beta$. Therefore, by 1.6.2 there is a homomorphism θ from (F,t) to (A,a) such that $s\theta=a_s$ for all $s\in S$.

Now β and $\theta\alpha$ are homomorphisms from (F,t) to (B,b) such that $s\theta\alpha=a_s\alpha=s\beta$ for all $s \in S$. Therefore, by the uniqueness part of the Theorem 1.6.2, we have $\theta\alpha=\beta$, as required.

1.6.4 <u>**THEOREM**</u>: Let (G,c) be a pointed-group with a generating set M. Let (F,t) be a free pointed-group of rank greater than or equal to the cardinality of M. Then (G,c) is isomorphic to a factor pointed-group of (F,t).

Proof: Let (F,t) be freely generated by S. Then F is generated by $S\cup\{t\}$. Now since the cardinality of M is less than or equal to the cardinality of S so there is an onto map α from S to M. Thus by Theorem 1.6.2 the function α yields a homomorphism α^* from (F,t) to (G,c) such that $\alpha^*|S = \alpha$. Since the image of α^* contains M and also M generates (G,c), it follows that α^* is an epimorphism from (F,t) to (G,c). Thus, by Theorem 1.4.3, we have $(G,c) \cong (F/R, tR)$ where R is the kernel of α^*. Thus the result follows:

From this theorem it follows that every finitely generated pointed-group is a factor pointed-group of a free pointed-group of finite rank (equal to the cardinality of a generating set of that pointed-group). More, explicitely, every n-generator pointed-group is a factor pointed-group of a free pointed-group of rank n. Clearly, this representation of a pointed-group (G,c) as a factor pointed-group of some free pointed-group (F,t) is by no means unique, because it depends on the choice of a generating set of (G,c). But we have:

1.6.5 <u>**COROLLARY**</u>: Let (G,c) be a finitely generated pointed-group. Then (G,c) is isomorphic to a factor pointed-group of a finitely generated free pointed-group (F,t) of any rank greater than or equal to the cardinality of the generating set of (G,c).

Now an immediate consequence of the Theorem 1.6.2 is:

1.6.6 <u>**COROLLARY**</u>: Let (F,t) be a free pointed-group with a free generating set S. Then any mapping α from S to F can be extended uniquely to an endomorphism α^* of (F,t) such that $\alpha^*|S = \alpha$.