FINITE BASIS PROBLEM FOR POINTED - GROUPS

By

ZAFAR ALI

Thesis submitted for the degree of Ph.D at the Punjab University, Lahore (Pakistan)

DECLARATION

The work presented in this thesis, is my own except where indicated.

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ABSTRACT

A pointed-group is an ordered pair (G,c) where G is a group and c is a specific element of G.Thus a pointed-group is a group together with a distinguished element. Pointed-groups may be regarded as a particular type of universal algebras.

The object of the thesis is to generalize ideas and results concerning varieties of groups to varieties of pointed-groups. For example, a law of a pointed-group (G,c) is a word w in the variables y, x_1, x_2 .... such that w takes the value 1 whenever c is substituted for y and arbitrary elements of G are substituted for x_1, x_2, ...

The principal topic studied is the *finite basis question*, which asks if the laws of (G,c) are all consequences of some finite set of laws. We prove that the laws of (G,c) are finitely based if G is either nilpotent or metabelian, thereby generalizing theorems of R.C. Lyndon and D.E. Cohn.

However, the main reasons for studying laws of pointed-groups is to provide a test case for a conjecture of Sheila Oates Macdonald which states that if $\underline{\underline{V}}$ is any variety of universal algebras whose congruence lattices are modular,

then every finite algebra in $\underline{\underline{V}}$ has a finite basis for its laws.

One well-known instance is a theorem of Sheila Oates and M.B. Powell, which states that the laws of every finite group are finitely based.

It is not difficult to see that the congruence lattice of any pointed-group is modular, therefore, we are led to the study of pointed-groups (G,c) where G is a finite group. In this case we prove that the laws of (G,c) are finitely based, under additional assumptions on G, for example that the Sylow sub groups of G are abelian. However, we have been able to prove that this is not always true. As we have found a finite pointed-group whose set of laws has not a finite basis thereby the generalization of the theorem of Sheila Oates and M.B. Powell to finite pointed-groups is false and thereby providing a counter example to the conjecture of Sheila Oates Macdonald to be false.

INTRODUCTION

The main object of this work is to generalize the idea of a law in a group. A law in a group is a word in the variables x_1, x_2, ... which takes the value 1 under every substitution of group element for x_1, x_2, ...

More generally, we shall study words in the variables y, x_1, x_2, ... under substitution of a fixed element of a group for y and arbitrary elements of a group for x_1, x_2, ... Thus we shall study ordered pairs (G,c) where G is a group and c is a distinguished element of G. Such ordered pairs will be called pointed-groups.

In chapter 0, we collect the background materials on groups and varieties of groups which we intend to generalize.

A pointed-group may be regarded as a group together with an additional nullary operation (picking out the distinguished element). Viewed in this way, pointed-groups are types of universal algebras. Thus we are able to generalize to pointed-groups many basic ideas and results concerning groups. This is done in chapter I.

In chapter II, we continue this programme and generalize to pointed-groups basic ideas and results concerning varieties of groups. In particular, a law of a pointed-group (G,c) is a word in variables y, x_1, x_2,.... which takes the value 1 and every substitution of c for y and arbitrary group elements for x_1, x_2, ...

Chapter III and IV are concerned with the finite basis question, which asks if the laws of (G,c) are consequences of some finite set of laws. This was the principal reason for the study of laws of pointed-groups. A conjecture of sheila Oates Macdonald states that if $\underline{V}$ is any variety of universal algebras whose congruence lattices are modular, then every finite algebra in $\underline{V}$ has finite basis for its laws. We shall see (Chapter IV) that the congruence lattice of any pointed-group is modular. Therefore, pointed-groups (G,c) where G is a finite group provide a test case for the conjecture and thereby generalizing the theorem of Sheila Oates and M.B. Powell.

Fortunately, although some partial results are obtained, we have been able to decide that not every finite pointed-group (G,c) where G is a finite group, has a finite basis for its laws. This has been shown in chapter V, where a counter example has been given.