

Chapter 0

PRELIMINARIES

PRELIMINARIES§ 0.1 GENERAL NOTE

The notation and definitions are standard and can be found in any standard book on group theory, however, in this work all the mappings are written on the right.

§ 0.2 VARIETIES OF GROUPS (BASIC FACTS AND DEFINITIONS)

The reader is referred to Hana Neumann's book [12] for basic results and definitions concerning varieties of groups. A detailed discussion of the results which we intend to generalize may be found in this book. For our purpose here we merely outline some of these results and some essential definitions of the concepts are stated below:

i. Free Groups

Let S be any set. The *absolutely free group* on S , denoted by F_S , consists of all formal expressions of the form

$$\begin{array}{cccc} m_1 & m_2 & & m_k \\ s_1 & s_2 & \dots & s_k \end{array}$$

where the m_i are non-zero integers and $s_i \neq s_{i+1}$ ($i=1, \dots, k-1$). The empty expression is denoted by 1. The product of the

two expressions $s_1^{m_1} s_2^{m_2} \dots s_k^{m_k}$ and $t_1^{n_1} t_2^{n_2} \dots t_r^{n_r}$ is the result obtained from $s_1^{m_1} s_2^{m_2} \dots s_k^{m_k} t_1^{n_1} t_2^{n_2} \dots t_r^{n_r}$ by combining adjacent factors where possible. For example, if $S = \{a, b\}$ we have

$$(a^2 b^{-1} a^{-3}) (a^3 b^2) = a^2 b^1$$

The definition of product can be made precise and gives F_S a group structure (see any book on group theory). Identifying s^1 with s for all $s \in S$, S becomes a subset of F_S and clearly S generates F_S . By its definition F_S has no non-trivial relations on S . The cardinality of S is called the rank of F_S . Thus $F_S \cong F_T$ if and only if $|S| = |T|$ i.e. if and only if S and T have the same cardinality.

ii. Verbal Sub group

It will be convenient to have some notation for a particular free group X of countable rank. X will denote the free group on a generating set consisting of symbols x_1, x_2, \dots . The symbols will be called variables. Any element of X will be called a word. Every word w has the form $x_{i_1}^{m_1} x_{i_2}^{m_2} \dots x_{i_k}^{m_k}$ and so contains only finitely many variables. Thus for some n , the variables of w belong to $\{x_1, x_2, \dots, x_n\}$. To illustrate this, w is some times written as

$$w = w(x_1, x_2, \dots, x_n)$$

Now if G is a group, $\alpha : X \rightarrow G$ a homomorphism and w a word, the image $w\alpha$ of w under α is called a value of w in G . If $w = x_{i_1}^{m_1} x_{i_2}^{m_2} \dots x_{i_k}^{m_k}$ and we write $x_i \alpha = g_i$ for each

i then

$$w\alpha = g_{i_1}^{m_1} g_{i_2}^{m_2} \dots g_{i_k}^{m_k}$$

Thus the value $w\alpha$ is obtained from w by substituting elements of G for the variables of w . Conversely every element $g_{i_1}^{m_1} g_{i_2}^{m_2} \dots g_{i_k}^{m_k}$ of G arising by substitution in this way is a value of w .

EXAMPLE: If g_1 and g_2 are elements of any group, then $[g_1, g_2]$ denotes the commutator $g_1^{-1} g_2^{-1} g_1 g_2$. Thus $[g_1, g_2]$ is a value of the word $[x_1, x_2]$.

Further if w is a word and G is a group, $w(G)$ denotes the sub group of G generated by all values of w in G . More generally, if W is a set of words, $W(G)$ denotes the sub group of G generated by all values of w of the elements of W . Thus $W(G)$ is called the *verbal-sub group* of G corresponding to W .

EXAMPLE: The derived group G' of a group G is the verbal-sub group of G corresponding to the word $[x_1, x_2]$.

Suppose $\alpha : G \rightarrow H$ is a homomorphism of groups and

let $w = w(x_1, x_2, \dots, x_n)$ be a word. Then for all $g_1, g_2, \dots, g_n \in G$ we have

$$w(g_1, g_2, \dots, g_n)\alpha = w(g_1\alpha, g_2\alpha, \dots, g_n\alpha)$$

Thus the set of values of w in $G\alpha$ is the image under α of the set of values of w in G . Hence if W is a set of words, the set of all values in G of the elements of W is the image under α of the set of all values in G of the elements of W . Thus $W(G)\alpha = W(G\alpha)$.

A sub group H of a group G is called *fully invariant* if $H\alpha \subseteq H$ for all endomorphism α of G . Thus every verbal-sub group $W(G)$ of a group G is fully invariant.

Now let W be any set of words. Then $W(X)$ is fully invariant sub group of X . By definition $W(X)$ is generated by element of the form $w\alpha$ where $w \in W$ and α is a homomorphism from X to X i.e. an endomorphism of X . Any fully invariant sub group of X which contains W must contain all elements $w\alpha$ and so must contain $W(X)$. Thus $W(X)$ is the smallest fully invariant sub group of X which contains W . We call $W(X)$ the *fully invariant closure* of W and write $W(X) = \overline{W}$. Note that for the group X , but not in general the converse of the result: that every verbal-sub group $W(G)$ of a group G is fully invariant holds, i.e. every fully invariant sub group is verbal. Thus if W is fully invariant sub group of X , then

$$W = \bar{W} = W(X).$$

iii. Varieties

If w is a word and G is a group, w is said to be a law of G if w takes only the trivial value in G i.e. if $w\alpha = 1$ for every homomorphism $\alpha : X \rightarrow G$. Thus w is a law of G if and only if $w(G) = \{1\}$.

EXAMPLE: $\{x_1, x_2\}$ is a law of every abelian group.

Let W be a set of words. The variety defined by W , denoted by $\text{var } W$ or \underline{W} , is the class of all groups for which every element of W is a law. Thus $G \in \text{var } W$ if and only if $W(G) = \{1\}$.

Next, if V is a set of words and w is a word, we shall say that w is a consequence of V if w is a law of every group of $\text{var } V$. Sets of words V and W are said to be equivalent if every element of V is a consequence of W and every element of W is a consequence of V . Clearly V and W are equivalent if and only if $\text{var } V = \text{var } W$.

EXAMPLE: i. $\{x_1, x_2\}$ is a consequence of x_1^2 i.e. a group in which every element has order 1 or 2 is abelian.

ii. $\{x_1^4, x_1^6\}$ is equivalent to $\{x_1^2\}$.

Another simple example of equivalence is that a word w is always equivalent to any word w which comes from w by a change of variables. For example $[x_1, x_2]$ is equivalent to $[x_2, x_3]$.

Moreover, let \underline{Y} be any class of groups. Then $Q\underline{Y}$ denotes the class of all groups isomorphic to a factor group of a group in \underline{Y} , $S\underline{Y}$ the class of all groups isomorphic to a sub group of a group in \underline{Y} and $C\underline{Y}$ the class of all groups isomorphic to a cartesian product of groups in \underline{Y} . We say that \underline{Y} is (Q,S,C)-closed if $Q\underline{Y} \subseteq \underline{Y}$, $S\underline{Y} \subseteq \underline{Y}$ and $C\underline{Y} \subseteq \underline{Y}$, i.e. if \underline{Y} is closed under the formation of factor groups, sub groups and cartesian products, and groups isomorphic to these.

Thus every variety is (Q,S,C)-closed. A remarkable theorem of Birkhoff states that any non-empty class of groups is a variety if and only if it is (Q,S,C)-closed.

Now for any non-empty class of groups \underline{Y} , $QSC\underline{Y}$ contains \underline{Y} and is a variety. Also every variety containing \underline{Y} must contain $QSC\underline{Y}$ because varieties are (Q,S,C)-closed. Thus $QSC\underline{Y}$ is called the variety generated by \underline{Y} and write $\text{var } \underline{Y}$. Now if \underline{Y} is the empty class we define $\text{var } \underline{Y}$ to be the variety of all groups of order 1 i.e. the variety defined by the word x_1 .

Now given a group G , the variety generated by G is defined to be the intersection of all varieties containing G .

By a group property of a variety of groups we mean a property which is satisfied by all the groups of that variety. For example, abelian varieties, nilpotent varieties and metabelian varieties etc. are varieties such that the groups of those varieties respectively are abelian, nilpotent and metabelian etc.

Now if a set of word W defines a variety \underline{V} then every law of \underline{V} is a consequence of W . So W is called a *basis* for the laws of \underline{V} . If W is a basis of \underline{V} then the other bases of \underline{V} are the sets of words equivalent to W .

Now a variety \underline{V} is said to be *finitely based* if it can be defined by a finite set of laws i.e. \underline{V} has a *finite basis* if all the laws of \underline{V} are a consequence of some finite set of laws.

One of the reason for interest in the finite basis problem is its connection with the free group X , because a basis for a variety \underline{V} is a set of words whose fully invariant closure in X is the set of all laws of \underline{V} . Thus a variety is finitely based if and only if the corresponding fully invariant sub group of X is the fully invariant closure of a finite set of words.

Now we mention ~~some~~ of the known results of finitely

based varieties as follows:

In finitely based varieties the most interesting case is that of a variety all of whose sub varieties are finitely based. Thus the first significant finite basis result was obtained by R.C. Lyndon in [10]. It states that for all c , the variety \underline{N}_c of all nilpotent groups of class at most c is *hereditarily finitely based*. (By hereditarily finitely based variety we mean a variety whose every sub variety is finitely based).

But probably the most interesting finite basis result is the theorem of Sheila Oates and M.B. Powell in [13] which states that the variety generated by a finite group is finitely based. In other words, the laws of a finite group are all consequences of a finite set.

Another finite basis result begins with the theorem of D.E. Cohn in [3] which states that variety $\underline{A} \underline{A}$ of all metabelian groups is hereditarily finitely based. Cohn's proof of this theorem introduced an important technique i.e. the use of well-quasi-ordered sets.

Vaughan Lee in [14] proved that for any c , $\underline{N}_c \underline{A} \cap \underline{A} \underline{N}_c$ is hereditarily finitely based, which is of course a common extension of R.C. Lyndon and Cohn results. However,

Vaughan Lee in [15] has shown that $\underline{N}_2\underline{N}_2$ has a sub variety which is not finitely based. Thus $\underline{N}_2\underline{N}_2$ is not hereditarily finitely based.

Moreover, one of the great importance in the proof of the Oates and Powell theorem in [13] is the idea of a *locally finite variety* i.e. a variety in which every finitely generated group is finite. Now as every variety is generated by its finitely generated group see [12]. Thus every locally finite variety is generated by its finite group.

A finite group G is called *critical* if G does not belong to the variety generated by its proper sub groups and proper factor groups.

Thus a group G is called *sub group critical* if G is finite and G does not belong to variety generated by its proper sub groups. For example, the dihedral group of order 6 is sub group critical as it is a non-abelian group with the property that all its proper sub groups are abelian. The significance of the sub group critical lies in the following result:

If \underline{V} is variety which is generated by its finite groups, then \underline{V} is generated by its sub group-critical groups. Thus every locally finite variety is generated by its sub group critical groups.

Locally finite varieties arise in the context of the

Oates Powell Theorem because of the fact:

The variety generated by any finite group is locally finite.

Now let n be a positive integer and let X_n denote the sub group of X generated by $\{x_1, x_2, \dots, x_n\}$. Clearly X_n is the free group on $\{x_1, x_2, \dots, x_n\}$. A word $w \in X_n$ is called an n -variable word. Now let \underline{V} be any variety and let $V = \underline{V}(X)$ be the set of all laws of \underline{V} . Then $V \cap X_n$ is the set of all laws of \underline{V} which belongs to X_n . Now for each n let $\underline{V}^{(n)}$ denote the variety defined by $V \cap X_n$ i.e. $\underline{V}^{(n)}$ is a variety defined by the set of n -variable laws of \underline{V} . Now the significance of the varieties $\underline{V}^{(n)}$ in the proof of the Oates Powell Theorem lies in the fact that for every locally finite variety \underline{V} , $\underline{V}^{(n)}$ is *finitely based* for each positive integer n . Now let A be a finite group and let $\underline{V} = \text{var } A$. Thus \underline{V} is *finitely based* if and only if (i) $\underline{V}^{(m)}$ is locally finite and (ii) $\underline{V}^{(m)}$ contains, upto isomorphism only finitely many sub group-critical groups.

Moreover, a finite group G is said to be *monolithic* if it has a unique (non-trivial) minimal normal sub group. Such a sub group of G is said to be *monolith* of a monolithic group G .

Thus a critical group is monolithic (by 51.32 of [12]).